Selected Definitions, Equations, and Theorems

Area Vector

The area vector of a flat, oriented surface is a vector $\vec{A}$ such that

- The magnitude of $\vec{A}$ is the area of the surface.
- The direction of $\vec{A}$ is the direction of the orientation vector $\vec{n}$.

Flux Through a Surface - Constant vector field, flat surface

Suppose the velocity vector field, $\vec{v}$, of a fluid is constant and $\vec{A}$ is the area vector of a flat surface. Then, the flux through the surface is given by $\vec{v} \cdot \vec{A}$.

Flux Through a Surface Given By a Graph

Suppose the surface $S$ is the part of the graph $z = f(x,y)$ above a region $R$ in the $xy$-plane, and suppose $S$ is oriented upward. The flux of $\vec{F}$ through $S$ is

$$\int_S \vec{F} \cdot d\vec{A} = \int_R \vec{F}(x,y,f(x,y)) \cdot (-f_z\vec{i}, -f_y\vec{j}, \vec{k}) dx dy$$

Flux Through a Cylinder

The flux of $\vec{F}$ through the cylindrical surface $S$, of radius $R$ and oriented away from the $z$-axis, is given by

$$\int_S \vec{F} \cdot d\vec{A} = \int_R \vec{R}(R,\Theta,z) \cdot (\cos\Theta\vec{i} + \sin\Theta\vec{j}) Rd\Theta, $$

where $T$ is the $\Theta z$-region corresponding to $S$.

Note: We must be careful using this formula if $S$ is oriented along the $x$- or $y$-axis. The vector field should be put in terms of $(f(y,z), y, z)$ or $(x, f(x,y), z)$, respectively. The normal vectors should be given by $(\cos\Theta\vec{j} + \sin\Theta\vec{k})$ or $(\sin\Theta\vec{i} + \cos\Theta\vec{j})$ respectively. Remember the right hand rule!
Flux Through a Sphere
The flux of $\vec{F}$ through the spherical surface $S$, with radius $R$ and oriented away from the origin, is given by

$$\int_S \vec{F} \cdot d\vec{A} = \int_S \vec{F} \cdot \frac{\vec{r}}{||\vec{r}||} dA$$

$$= \int_T \vec{F}(R, \Theta, \phi) \cdot (\sin \phi \cos \Theta \vec{i} + \sin \phi \sin \Theta \vec{j} + \cos \phi \vec{k}) R^2 \sin \phi d\phi d\Theta,$$

where $T$ is the $\Theta\phi$-region corresponding to $S$.

Note: Some students have asked what this $\vec{r}$ vector is. The book and WebAssign both like to use it, but they didn’t introduce the notation very explicitly. It is what we might call the position vector, given by $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$.

Divergence of a Vector Field
The divergence, also known as flux density, of a smooth valued vector field $\vec{F}$, written $\text{div} \, \vec{F}$, is a scalar-valued function determined as follows: If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\text{div} \, \vec{F}(x, y, z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

The Divergence Theorem
If $W$ is a solid region whose boundary $S$ is a piecewise smooth surface, and if $\vec{F}$ is a smooth vector field on an open region containing $W$ and $S$, then

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \text{div} \, \vec{F} \, dV,$$

where $S$ is given the outward orientation. The Curl of a Vector Field
The curl of a smooth vector field $\vec{F}$, written $\text{curl} \, \vec{F}$, is the vector field given as follows: if $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\text{curl} \, \vec{F} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \vec{i} + \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \vec{j} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \vec{k}$$
Stokes’ Theorem
If $S$ is a smooth oriented surface with piecewise smooth, oriented boundary $C$, and if $\vec{F}$ is a smooth vector field on an open region containing $S$ and $C$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \text{curl} \vec{F} \cdot d\vec{A}$$

Gradient and Curl
Recall that gradient fields are path independent. In particular, the circulation around a closed curve is always 0 on gradient fields, by the Fundamental Theorem of Calculus for Line Integrals. Thus, we see that $\text{curl} \text{grad} f = \vec{0}$. Thus, if $\vec{F} = \text{grad} f$, then $\text{curl} f = 0$. As it turns out, the converse is also true. This motivates the next concept.

The Curl Test for Vector Fields in 3–Space
Suppose $\vec{F}$ is a smooth vector field on 3-space such that

- The domain of $\vec{F}$ has the property that every closed curve in it can be contracted to a point in a smooth way, staying at all times within the domain.
- $\text{curl} \vec{F} = \vec{0}$

Then $\vec{F}$ is path independent, and thus $f$ is a gradient field.

Divergence Test for Vector Fields in 3–Space
Suppose $\vec{F}$ is a smooth vector field on 3-space such that

- The domain of $\vec{F}$ has the property that every closed surface in it is the boundary of a solid region completely contained in the domain.
- $\text{div} \vec{F} = 0$

Then $\vec{F}$ is a curl field.