Definition 0.1. $S$ is a semi-algebra if

1. $\emptyset, \Omega \in S$
2. closed under finite intersection
3. $S \subseteq S \Rightarrow S^c = \text{finite disjoint union of sets in } S$.

An example is 
$$S_1 = \{\emptyset, \mathbb{R}, (a, b) \text{ for } -\infty \leq a < b < \infty, (a, \infty) \text{ for } a \in \mathbb{R}\}.$$ 

Lemma 1. Let $S$ be a semi-algebra. Then, 
$$\bar{S} = \text{finite disjoint unions of sets in } S$$ 

is an algebra.

Proof. This is Lemma 1.1.3 in Durrett. It is enough to show $\bar{S}$ is closed under finite intersection and complements. Intersections: Let $A, B \in S$. Then, $A = \cup_i A_i$ and $B = \cup_j B_j$ are both finite disjoint unions of sets $A_i$ and $B_j$ in $S$. Note that $A_i \cap B_j \in S$ by definition. Now, write $A \cap B = \cup_{i,j} A_i \cap B_j$ which is a finite disjoint union of sets in $S$ and therefore in $\bar{S}$.

Complements: Let $A \in \bar{S}$ and write $A = \cup_i A_i$ as a finite disjoint union of sets in $S$. Then, $A^c = \cap_i A_i^c$. Now, $A_i^c \in \bar{S}$ by definition of $S$. Since, we have just shown that $\bar{S}$ is closed under finite (using induction) intersection, $A^c \in \bar{S}$.

Lemma 2. Let $\mu_0$ be a set function on semi-algebra $S$ such that (i) $\mu_0(A) \geq \mu_0(\emptyset) = 0$ for $A \in S$ and (ii) $\mu_0(A) = \sum_{i=1}^n \mu_0(A_i)$ when $A \in S$ and $A$ can be written as a finite disjoint union of sets $A_i$ in $S$.

Define $\bar{\mu}$ on $\bar{S}$, the algebra generated by $S$ in Lemma 1, by 
$$\bar{\mu}(A) = \sum_{i=1}^n \mu_0(A_i)$$

where $A \in \bar{S}$ is written $A = \cup_{i=1}^n A_i$ as a disjoint union of sets $A_i \in S$.

Then, $\bar{\mu}$ is well defined and finitely additive on $\bar{S}$.

Proof. This is part (a) of Lemma 1.1.5. First, $\bar{\mu}$ is well defined: Suppose that $A \in \bar{S}$ can be written another way, that is $A = \cup_{j=1}^m B_j$ with respect to disjoint sets $B_j$ in $S$. We need to check that $\sum_i \mu_0(A_i) = \sum_j \mu_0(B_j)$. Now, since $A_i, B_j \subseteq A$, we can write $A_i = A_i \cap A = \cup_{j=1}^m A_i \cap B_j$ and $B_j = \cup_{i=1}^n A_i \cap B_j$, which are both disjoint finite unions of sets in $S$ (as $S$ is closed to intersection). Hence, by properties of $\mu_0$,
$$\sum_i \mu_0(A_i) = \sum_i \sum_j \mu_0(A_i \cap B_j) = \sum_j \mu_0(B_j).$$

Finitely additive: Suppose $A \in \bar{S}$ and $A = \cup_j B_j$ is a finite disjoint union of sets $B_j$ in $\bar{S}$. Then, since each $B_j = \cup_i B_{i,j}$ is a finite disjoint union of sets $B_{i,j} \in S$. Hence, $A = \cup_{i,j} B_{i,j}$ which is a finite disjoint union of sets in $S$, and so $\bar{\mu}(A) = \sum_{i,j} \mu_0(B_{i,j}) = \sum_j \bar{\mu}(B_j)$. 


Definition 0.2. \( \mu \) is a premeasure on an algebra \( \mathcal{A} \) if

\begin{enumerate}
\item \( \mu(A) \geq \mu(\emptyset) \geq 0 \) for \( A \in \mathcal{A} \).
\item \( \mu \) is countably additive on \( \mathcal{A} \): if \( A, \{A_i\} \in \mathcal{A} \) and \( \{A_i\} \) disjoint such that \( A = \bigcup A_i \), then \( \mu(A) = \sum \mu(A_i) \).
\end{enumerate}

We say a premeasure \( \mu \) on an algebra \( \mathcal{A} \) is \( \sigma \)-finite if \( \Omega = \bigcup A_i \) where \( \{A_i\} \in \mathcal{A} \) and \( \mu(A_i) < \infty \).

Theorem 0.3 (Caratheodory). Let \( \mu \) be a premeasure on an algebra \( \mathcal{A} \). Then, \( \mu \) has an extension to a measure \( \nu \) on \( \sigma(\mathcal{A}) \). Also, if \( \mu \) is \( \sigma \)-finite on \( \mathcal{A} \), then \( \nu \) is unique.

Remark 0.4. Now, given a Stieltjes function \( F \) on \( \mathbb{R} \) and \( \mu_0((a,b]) = F(b) - F(a) \), we can build, via Lemma 2, a finitely additive, nonnegative set function on \( \mathcal{S}_1 \) (check). Then, using compactness ideas, one can extend this set function to a premeasure on \( \mathcal{S}_1 \). Finally, Caratheodory’s theorem allows to extend further to the Borel sets on \( \mathbb{R} \), that is the \( \sigma \)-field generated by \( \mathcal{S}_1 \).