Math 464 - Fall 10 - Homework 3

1. The probability mass function of a discrete RV $X$ is given in the table below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>0.1</td>
<td>0.2</td>
<td>0.15</td>
<td>0.2</td>
<td>0.15</td>
<td>0.05</td>
<td>0.05</td>
<td></td>
</tr>
</tbody>
</table>

Compute the following:
(a) the probability $X$ is even
(b) the probability that $1 \leq X \leq 8$
(c) the probability that $X$ is $-3$ given that $X \leq 0$.
(d) the probability that $X \geq 3$ given that $X > 0$.

2. Let $X$ be a discrete RV with the geometric distribution.
   (a) Compute $P(X > n)$.
   (b) Show that $P(X > n + k|X > n) = P(X > k)$.
   (c) (Exposition) The geometric distribution may be realized by flipping a biased coin until we get heads. ($X$ is the total number of flips needed.) Explain why the equation in part (b) is “obvious” in this experiment.

3. (a) Compute the mean and variance of the geometric distribution.
   (b) Compute the variance of the binomial distribution. Hint: It is easier to compute $E[X(X - 1)]$ than $E[X^2]$. Note that $E[X(X - 1)] = E[X^2] - E[X]$.

4. (Exposition) Let $X$ be a discrete RV whose range is $0, 1, 2, 3, \ldots$. Prove that
   \[ E[X] = \sum_{k=0}^{\infty} P(X > k) \]

5. We roll a six-sided die $n$ times. Each time the die comes up 1, we flip a fair coin. Let $X$ be the number of heads we get. Note that the number of times the coin is flipped is random.
   (a) Find the mean of $X$.
   (b) The probability mass function of $X$ is in our catalog. What is it? You do not have to prove your answer. If you get stuck, try working out the answer for a few small values of $n$.

6. (Exposition) Let $X$ be the number of eggs laid by an insect. We suppose that $X$ is a Poisson random variable with parameter $\lambda$. Each egg produces
an insect with probability $p$. The eggs are independent of each other. Let $Y$ be the number of insects that hatch from the $X$ eggs. Show that $Y$ is also a Poisson random variable and find its parameter (in terms of $\lambda$ and $p$). Hint: Use the partition theorem:

$$P(Y = k) = \sum_{n=k}^{\infty} P(Y = k|X = n)P(X = n)$$

If we are given that $X = n$, then $Y$ is a binomial RV.

7. We have two experiments with probability spaces $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$. The experiments are independent. $X_1$ is a discrete RV for experiment 1, and $X_2$ is a discrete RV for experiment 2. Let $p \in [0, 1]$, and suppose we have a coin with probability $p$ of heads. We define a new experiment and discrete RV $X$ as follows. We flip the coin. If it is heads, we perform experiment 1, and let $X$ be the value we get for $X_1$. If the coin is tails we perform experiment 2 and let $X$ be the value we get for $X_2$. Let $f_1(x)$ and $f_2(x)$ be the probability mass functions for $X_1$ and $X_2$. What is the probability mass function for $X$? Your answer should involve $f_1, f_2$ and $p$. 