Math 520a - Homework 0

You do not have to turn in the first 6 problems.

1. For $A \subset \mathbb{C}$, define $int(A) = \{z : \exists \epsilon > 0$ such that $D_\epsilon(z) \subset A\}$.
   (a) Prove $int(A)$ is open.
   (b) Prove that if $U$ is open and $U \subset A$, then $U \subset int(A)$.

2. Define $\bar{A} = \{z : \exists z_n \in A$ such that $z_n \rightarrow z\}$.
   (a) Prove $\bar{A}$ is closed.
   (b) Prove that if $F$ is closed and $A \subset F$, then $\bar{A} \subset F$.
   (c) Prove $\bar{A} = (int(A^c))^c$.

3. For $A \subset \mathbb{C}$ define $\partial A = \bar{A} \setminus int(A)$. Prove that
   \[
   \partial A = \{z : \exists z_n \in A$ with $z_n \rightarrow z$, and $\exists w_n \notin A$ with $w_n \rightarrow z\}
   \]

4. The complex exponential: One way to define $e^z$ for complex $z$ is by its power series. Here is another. Letting $z = x + iy$, we should have
   \[
e^z = e^{x+iy} = e^x e^{iy} = e^x [\cos(y) + i \sin(y)]\]
   So we can define $e^{x+iy}$ to be the complex valued function whose real part is $u(x + iy) = e^x \cos(y)$ and whose imaginary part is $v(x + iy) = e^x \sin(y)$.
   (a) Prove this is an entire function and it satisfies the differential equation $(e^z)' = e^z$.
   (b) Let $a$ be real and let $C$ be the vertical line given by $Re(z) = a$. What is the image of $C$ under the map $e^z$?

5. Let $\Omega$ be the complex plane with the ray $(-\infty, 0]$ on the real axis removed:
   \[
   \Omega = \mathbb{C} \setminus \{z : Im(z) = 0, Re(z) \leq 0\}
   \]
   Any $z \in \Omega$ can be written uniquely as $re^{i\theta}$ with $-\pi < \theta < \pi$, $r > 0$. Define $\ln(z)$ to be $\ln(r) + i\theta$. Prove that $\ln(z)$ is analytic on $\Omega$ and that $e^{\ln(z)} = z$.

6. Define $\Omega$ as in the previous problem. The square root can be defined by $\sqrt{z} = \exp(\ln(z)/2)$. Let $\mathbb{H}$ be the upper half of the complex plane (not including the real axis).
   (a) What is the image of $\Omega$ and of $\mathbb{H}$ under the map $\sqrt{z}$?
The following two problems should be turned in.

7. Let \( f(z) \) be defined on a neighborhood of \( z_0 \). Suppose there is a complex number \( w \) such that for all angles \( \theta \),

\[
\lim_{r \to 0^+} \frac{f(z_0 + re^{i\theta}) - f(z_0)}{re^{i\theta}} = w
\]

Does it follow that \( f \) is complex differentiable at \( z_0 \)? Prove that it does or give a counterexample. In the above \( r \) goes to 0 only through positive real numbers.

8. Let \( f(z) = \sqrt{1 - z^2} \) with \( \sqrt{\cdots} \) defined as in previous problem. What is the image of the upper half plane \( \mathbb{H} \) under \( f \) ?