1. Is it possible to define a branch of the logarithm \( f(z) \) such that for all positive integers \( n \), \( f(n) = \log(n) + 2\pi i n \)? You should justify your answer, i.e., show it cannot be done or show how to do it.

2. Let
\[
f(z) = \frac{1}{2}(z + \frac{1}{z})
\]
Let \( U = \{z \in \mathbb{H} : |z| > 1\} \). Show that \( f \) is a conformal map of \( U \) onto the upper half plane \( \mathbb{H} \).


4. Let \( U \) be a simply connected region which is not empty and not the entire plane. Let \( z_0 \in U \).
   (a) Prove there is a unique \( r > 0 \) such that there is a conformal map \( f \) from \( U \) onto the disc with radius \( r \) centered at the origin satisfying \( f(z_0) = 0 \), \( f'(z_0) = 1 \). The radius \( r \) is called the conformal radius of \( U \) (with respect to \( z_0 \)). We will denote it by \( r(U, z_0) \).
   (b) Let \( U_1 \) and \( U_2 \) be simply connected regions which are not empty and not the entire plane. Suppose that \( U_1 \subset U_2 \) and \( z_0 \in U_1 \). Prove that \( r(U_1, z_0) \leq r(U_2, z_0) \).

5. Suppose that \( f \) is analytic on the annulus \( \{z : \rho_1 < |z| < \rho_2\} \). From what we did in class we know that it has a Laurent series of the form
\[
f(z) = \sum_{n=-\infty}^{\infty} a_n z^n
\]
meaning that the series converges to \( f(z) \) on the annulus. Moreover the convergence is absolute.

Define
\[
R_1 = \limsup_{n \to \infty} |a_{-n}|^{1/n}
\]
\[
\frac{1}{R_2} = \limsup_{n \to \infty} |a_n|^{1/n}
\]
(a) Prove that $R_1 \leq \rho_1$ and that $R_2 \geq \rho_2$.

(b) Prove that the Laurent series converges absolutely on $\{z : R_1 < |z| < R_2\}$ and uniformly on compact subsets of this set, and so defines an analytic continuation of $f$ to this annulus.

(c) Prove that if $f$ has an analytic continuation to an annulus $\{z : r_1 < |z| < r_2\}$ with $r_1 \leq R_1$ and $r_2 \geq R_2$, then $r_1 = R_1$ and $r_2 = R_2$. In other words the annulus in part (b) is the largest annulus (about 0) containing the original annulus on which $f$ has an analytic continuation.

6. Moebius transformations give homeomorphisms of the Riemann sphere. Find all Moebius transformations that corresponds to rotations of the sphere.

7. Problem 1 on page 108 of the book leads us through a proof of the famous Koebe 1/4 theorem. (The book calls it the Koebe-Bierbach theorem.) Do at least parts (a) and (b).