1. **Polarization identity.** Prove that in any complex inner product space \((x, y) = 1/4[\|x + y\|^2 - \|x - y\|^2 + i(\|x + iy\|^2 - \|x - iy\|^2)]\).

2. **Jordan-von Neumann theorem.** In an arbitrary normed vector space the formula of problem 1 does not necessarily define an inner product. Prove that it does if and only if the norm satisfies the parallelogram law:
\[
\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).
\]

3. **Projection onto a finite-dimensional subspace.** Let \(X\) be an inner product space and let \(x_1, \ldots, x_N \in X\) be an orthonormal set. Prove that the expression
\[
\|x - \sum_{n=1}^{N} c_n x_n\|
\]
attains its minimum for \(c_n = (x, x_n)\). Interpret this result as an orthogonal decomposition. Do we need to assume \(X\) is complete? Why or why not?

4. **Finding the closed linear span.** Let \(\alpha\) be a complex number with \(0 < |\alpha| < 1\). In the space \(l^2\) we consider the vectors
\[
x_n = (1, \alpha^n, \alpha^{2n}, \ldots, \alpha^{kn}, \ldots).
\]
Find the closed linear span of the \(x_n, n = 1, 2, \ldots\).

5. **Fourier series of square integrable functions.**
   a) Prove that the family of harmonics \(e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}, n = 0, \pm 1, \pm 2, \ldots\) is an orthonormal basis in \(L^2[-\pi, \pi]\). One way to do it is to use Weierstrass approximation theorem.
   b) Use the Fourier expansion of the function \(x^2\) to prove that
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]

6. **Gram-Schmidt orthonormalization, Legendre polynomials.**
   a) Apply the Gram-Schmidt process to the functions \(1, x, x^2, x^3, \ldots\) on the interval \([-1, 1]\) with the \(L^2\) inner product. The resulting polynomials are the Legendre polynomials, \(\phi_n(x)\). Calculate the first four of them.
   b) Prove that
\[
\phi_n(x) = \sqrt{\frac{2n + 1}{2 \cdot 2^n n!}} \frac{d^n}{dx^n}(x^2 - 1)^n
\]
for all \(n\).
   c) Prove that \(\phi_n, n = 0, 1, \ldots\) is a basis in \(L^2[-1, 1]\).

7. **Hermite functions.** In the space \(L^2(\mathbb{R})\) (with the Lebesgue measure) consider vectors \(\psi_0, \psi_1, \ldots\), where \(\psi_0(x) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}x^2}\) and for \(n \geq 1\)
\[
\psi_n(x) = (2^n n!)^{-\frac{1}{2}} \pi^{-\frac{n}{2}} e^{-\frac{1}{2}x^2} \frac{d^n}{dx^n}(e^{-x^2}).
\]
a) Prove that $\psi_n$ form an orthonormal set.

b)* Prove that this orthonormal set is complete, i.e. that it is an orthonormal basis.

8*. **Stability of orthonormal bases.** Let $\mathcal{H}$ be a Hilbert space with a countable orthonormal basis $x_1, x_2, \ldots$. Let $y_1, y_2, \ldots$ be another orthonormal set, such that

$$\sum_{n=1}^{\infty} \| x_n - y_n \|^2 < \infty.$$ 

Prove that this set is also a basis, i.e. that its linear span is all of $\mathcal{H}$.

9. **Hilbert space geometry.** In a Hilbert space $H$, let $\|x - x_1\| = R$ and $\|x - x_2\| = R$ be two spheres with $\|x_1\| = \|x_2\|$. Show that the intersection of the two spheres is a sphere. Find its radius and its center. Show that it lies in an affine subspace of the form $v + Y$, where $v \in H$ and $Y$ is the subspace orthogonal to the vector $x_1 - x_2$.

10**. **Are Hilbert spaces the only self-dual Banach spaces?** Let $X$ be a real Banach space. Suppose there exists a linear isomorphism $T : X \to X'$, which is an isometry: $|Tx| = |x|$ for every $x$, where $|Tx|$ denotes the norm (of the linear functional) in the dual space. Does it follow that $X$ is a Hilbert space, i.e. that the norm in $X$ comes from an inner product? I do not know the answer to this question.