Mindful Manipulation: What Algebra Do Students Need for Calculus?

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10. Simplify completely: \(2u(3u^2 - 1) - (-8u^3 - 14u + 6)\).

11. Simplify completely: \(4(2x + 1)^2 + 3(2x + 1) + 1\).

12. Factor completely: \(32x^4y - 162y\).
2. Simplify as much as possible:

\[ \frac{5n - 10}{4 - n^2} \quad \text{Assume } n \neq \pm 2. \]

3. Perform the indicated operation and simplify as much as possible:

\[ \frac{2 - 3t}{\sqrt{t + 2}} + 3\sqrt{t + 2} = \]

4. Perform the indicated operation and simplify as much as possible.

\[ \frac{2(y + 1)^{1/2} - y(y + 1)^{-1/2}}{y + 1} = \]
Don’t just do something, stand there!

\[ \int \left( \frac{y^2 - 1}{y} \right)^2 dy, \quad \int \frac{t^2 + t}{\sqrt{t + 1}} dt, \quad \int \frac{1 - t}{t^2 - 2t} dt \]

\[ \int \sin^4 x dx, \quad \int \sin w \cos^4 w dw \]

What is the “outside function” in

\[ h(x) = (f(x))^3? \]

The skill needed for these problems is not only manipulation, but contemplation.
Algebraic foresight

\[
\lim_\limits{h \to 0} \frac{(x + h)^n - x^n}{h} = \frac{n(n + 1)(2n + 1)}{6}
\]

\[
\frac{d}{dx} \left( \arctan \left( \frac{1}{x} \right) \right) = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}
\]
What students see after calculus

\[ P(1 + \frac{r}{12})^{12n} \]

\[ L_0 \sqrt{1 - \left(\frac{v}{c}\right)^2} \]

\[ \frac{\sigma}{\sqrt{n}} \]
What is the Cause of This Error?

\[
\int \frac{1}{2x^2 + 4x + 4} \, dx = \int \frac{1}{x^2 + 2x + 2} \, dx
\]

\[
= \int \frac{1}{(x + 1)^2 + 1} \, dx
\]

\[
= \arctan(x + 1) + C
\]

Difference between expressions and equations

\[
2x^2 + 4x + 4
\]

\[
2x^2 + 4x + 4 = 0.
\]
The joy of symbols

- The symbols of algebra: $x, y, t$ (possibly $a, b, c$)
- The symbols of calculus: $x_0, \lim_{h \to 0}, \infty, \sum_{k=1}^{N}, \theta, \omega, \int_{a}^{b}$
- “Your students can’t even integrate $e^x$
- What we teach:
  $$\int e^{2x} \, dx$$
- What engineers want:
  $$\int e^{\omega t} \, dt$$
Implicit differentiation

If \( x^3 + y^3 - xy^2 = 5 \), find \( \frac{dy}{dx} \).

At pressure \( P \) atmospheres, a certain fraction \( f \) of a gas decomposes. The quantities \( P \) and \( f \) are related, for some positive constant \( K \), by the equation

\[
\frac{4f^2P}{1 - f^2} = K.
\]

1. Find \( df/dP \).
2. Show that \( df/dP < 0 \) always. What does this mean in practical terms?
The function \( y = A \sin \left( \left( \sqrt{\frac{k}{m}} \right) t \right) \) represents the oscillations of a mass \( m \) at the end of a spring. The constant \( k \) measures the stiffness of the spring.

1. Find a time at which the mass is farthest from its equilibrium position. Find a time at which the mass is moving fastest. Find a time at which the mass is accelerating fastest.

2. What is the period, \( T \), of the oscillation?

3. Find \( \frac{dT}{dm} \). What does the sign of \( \frac{dT}{dm} \) tell you?
A spherically symmetric vector field can be written in terms of the spherical coordinate $\rho = |\vec{r}|$ as

$$\vec{F} = f(\rho)\hat{e}_\rho.$$ 

Show that

$$\text{div} \vec{F} = \frac{1}{\rho^2} \frac{d}{d\rho} (\rho^2 f(\rho)) \ , \ \rho \neq 0.$$
Since $\vec{e}_\rho$ is a unit vector pointing radially away from the origin, $\vec{e}_\rho = \vec{r}/|\vec{r}| = \vec{r}/\rho$. Thus, we have

$$\vec{F} = \frac{f(\rho)}{\rho} \vec{r} = \frac{f(\rho)}{\rho} x \vec{i} + \frac{f(\rho)}{\rho} y \vec{j} + \frac{f(\rho)}{\rho} z \vec{k}.$$  

Let $g(\rho) = f(\rho)/\rho$. Since $\rho = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$, and $\partial \rho / \partial x = \frac{1}{2} (x^2 + y^2 + z^2)^{-1/2} \cdot 2x = x / \sqrt{x^2 + y^2 + z^2}$, using the chain rule, we have

$$\frac{\partial}{\partial x} g(\rho) = \frac{d}{d \rho} g(\rho) \cdot \frac{\partial \rho}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} g'(\rho),$$

$$\frac{\partial}{\partial y} g(\rho) = \frac{y}{\sqrt{x^2 + y^2 + z^2}} g'(\rho),$$

$$\frac{\partial}{\partial z} g(\rho) = \frac{z}{\sqrt{x^2 + y^2 + z^2}} g'(\rho).$$
What algebra do students need for calculus?

- Manipulative skill
- Observational skill
- Strategic competence
- Flexibility with the use of symbols
- Perception of structure
- A sense of purpose