1. Let $T$ be 3-dim linear vector space, $T^*$ - the dual space. The basis in $T$ is comprised by the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ with components

$\vec{e}_1 = -1, 1, 1,$

$\vec{e}_2 = 1, -1, 1,$

$\vec{e}_3 = 1, 1, -1.$

Find the dual basis $\vec{e}^i, i = 1, 2, 3$ in $T^*$.

Reminder: Vectors of the dual basis are defined by the condition:

$(\vec{e}_i, \vec{e}^j) = \delta_i^j$

The scalar product of the vector $A^i$ and the covector $B_j$ is defined as usual:

$(\vec{A}\vec{B}) = A^iB_j$

2. Let $x, y$ be coordinates on the plane $R^2$, and

$u = u(x, y),$
\[ v = v(x,y) \]

be the new curvilinear coordinates.

We will call these coordinates orthogonal, if

\[
(\nabla u, \nabla v) = \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} = 0.
\]

Suppose, the function \( u(x,y) \) is known. Can you find another function \( v(x,y) \)?

3. Let \( u^a \) be a vector, and \( S_{ab} \) - a family of numbers. Suppose that

\[
\phi = S_{ab} u^a u^b,
\]

and \( \phi \) is a scalar. Prove that \( S_{ab} \) is a tensor.

4. Let \( T_{a_1 \cdots a_n} \) be a tensor.

\( T_{(a_1 \cdots a_k) a_{k+1} \cdots a_n} \) means symmetrization over indexes \( a_1 \ldots a_k \).

\( T_{[a_1 \cdots a_k] a_{k+1} \cdots a_n} \) means alternization over \( a_1 \ldots a_k \).

Prove that if

\[
T_{(ab)c} = 0,
\]

then

\[
T_{[abc]} = 1/3 \left( B_{abc} + B_{bca} + B_{cab} \right).
\]

5. Let \( R^3 \) be an Euclidean space.

\[
\epsilon_{abc} = \begin{cases} 
1 & \sigma(a, b, c) = 0 \\
-1 & \sigma(a, b, c) = 1 
\end{cases}
\]

Prove that

\[
A. \quad \epsilon_{abc} \epsilon_{lmn} = \begin{vmatrix} 
\delta_{al} & \delta_{am} & \delta_{an} \\
\delta_{bl} & \delta_{bm} & \delta_{bn} \\
\delta_{cl} & \delta_{cm} & \delta_{cn} 
\end{vmatrix}
\]
B. \( \epsilon_{abc} \epsilon_{imc} = \delta_{al}\delta_{bm} - \delta_{am}\delta_{bl} \)

C. \( \epsilon_{abc} \epsilon_{lbc} = 2\delta_{al} \)

D. \( \epsilon_{abc} \epsilon_{abc} = 6 \)

6. Let \( F_{ij} \) be components of a twice-covariant antisymmetrical tensor (2-form \( F \)); let \( g_{ij} \) be a metric tensor. Find components of the following 1-form \( \omega \):

\[
\omega = \star d \star F
\]

Consider specially the simplest cases

\( g_{ij} = \delta_{ij} \)

7.

A star is seen in a laboratory frame under the angle \( \theta \). Under what angle it is seen by an observer, moving along the \( x \)-axis with velocity \( v \)?

8. A particle moves along the \( x \)-axis. Its acceleration is constant in time in the frame, where the particle velocity is zero. Find the world line of the particle, \( x = x(t) \).

9. Let \( R_{1,3}^3 \) be a pseudoeuclidean space with the metrics

\[
l^2 = x_0^2 - x_1^2 - x_2^2.
\]

Prove that any vector, tangent to this surface, is the space-like. (Its length is imaginary)

10. The distance between the Earth and the Sun is \( 1.5 \times 10^{13} \text{sm} \). Five minutes ago (according to the Earth-based clock) an eruption happens on
the Sun. In what reference frame this event will happen in one minute in the future?

11. Two elementary particles move towards each other. The velocity of the particles is $v$. The life time of each particle is $\tau$. How long one particle will live in the reference frame attached to the second particle?