Lecture 4

Solitons for shallow water

Let us consider a layer of ideal fluid of thickness $h$. The bottom is solid, the surface is free. Gravitational acceleration $g$ is perpendicular to the bottom, surface tension with coefficient $\sigma$ is included. The fluid is incompressible, density is equal to unity.

![Fluid domain](image)

Figure 4.1: Fluid domain.

Surface elevation is $\eta = \eta(x,t)$. Flow of fluid is potential

$$v = \nabla \Phi \quad \text{and} \quad \nabla \cdot v = 0$$

due to incompressibility, potential satisfies the Laplace equation

$$\Delta \Phi = 0 \quad (4.1)$$
Let us denote
\[ \Phi|_{y=\eta} = \Psi(x, t) \]  
(apparently)
\[ \frac{\partial \Phi}{\partial y} \bigg|_{y=-h} = 0 \]  
(Boundary conditions (4.2), (4.3) define uniquely a solution of the Laplace equation. Thus it is enough to follow the evolution of \( \eta(x, t) \), \( \Psi(x, t) \))

We will not prove following theorem. (The proof is put in application.)

**Theorem 1**
\( \eta, \Psi \) is a pair of canonical variables They obey equations
\[ \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \Psi} \]
\[ \frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta} \]  
(here \( H \) – total energy of fluid, consisting of kinetic and potential energy)
\[ H = T + U \]  
(Potential energy can be found in explicit form)
\[ U = \frac{1}{2} g \int_{-\infty}^{\eta} \eta^2 dx + \sigma \int (\sqrt{1 + \eta_x^2} - 1)dx \]  
(Kinetic energy is given by formula)
\[ T = \frac{1}{2} \int_{-\infty}^{\eta} \int_{-\infty}^{+\infty} (\nabla \Phi)^2 dydx = -\frac{1}{2} \int \Psi \cdot \Psi_n ds \]  
(Here \( \Psi_n \) is normal derivative of potential.)
\[ \Psi_n = \int G(s, s') \Psi(s')ds' \]  
(Here \( G(s, s') = G(s', s) \) – Green’s function for the Dirichlet-Neumann boundary problem. It cannot be expressed in an explicit form for arbitrary \( \eta(x, t) \).)
However, the Laplace equation can be solved approximately if $k\eta \to 0$, $k$ – characteristic wave number.

$$\Delta \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

$$\Phi = \Phi_0 + \Phi_1 + \ldots$$

$$\frac{\partial^2 \Phi_0}{\partial y^2} = 0$$

$$\Phi_0 = \Psi(x, t)$$

\begin{equation}
\frac{\partial^2 \Phi_1}{\partial y^2} = -\frac{\partial^2 \Psi}{\partial x^2} \quad (4.10)
\end{equation}

$$\Phi_1|_{y=\eta} = 0 \quad \frac{\partial \Phi_1}{\partial y}|_{y=-h} = 0$$

$$\Phi_1 = -\frac{1}{2} \frac{\partial^2 \Psi}{\partial x^2} \cdot y^2 + C_1 y + C_2$$

$$-\frac{1}{2} \frac{\partial^2 \Psi}{\partial x^2} \eta^2 + C_1 \eta + C_2 = 0$$

$$C_1 = -h \frac{\partial^2 \Psi}{\partial x^2}$$

$$C_2 = \frac{\partial^2 \Psi}{\partial x^2} \left( -\frac{1}{2} \eta^2 - h \eta \right)$$

Assuming that steepness of the surface is small $k\eta \to 0$, one can put $C_2 = 0$ because we need only derivatives of $\Phi$.

Thus

$$\Phi_1 = -\frac{\partial^2 \Psi}{\partial x^2} \left( \frac{1}{2} y^2 + hy \right)$$

$$\frac{\partial \Phi}{\partial x} \simeq -\frac{\partial^2 \Psi}{\partial x^2} \frac{\partial^3 \Psi}{\partial x^3}$$

$$\frac{\partial \Phi}{\partial y} \simeq -\frac{\partial^2 \Psi}{\partial x^2} (h + y)$$

$$\left( \frac{\partial \Phi}{\partial x} \right)^2 \simeq \frac{\left( \partial \Psi / \partial x \right)^2}{\left( \partial \Psi / \partial x \right)^2} - 2 \frac{\partial \Psi}{\partial x} \frac{\partial^2 \Psi}{\partial x^3} \left( \frac{1}{2} y^2 + hy \right) \simeq$$

$$\simeq \left( \frac{\partial \Psi}{\partial x} \right)^2 + 2 \left( \frac{\partial^2 \Psi}{\partial x^2} \right)^2 \left( \frac{1}{2} y^2 + hy \right)$$

16
\[
\int_{-h}^{\eta} \left[ y^2 + 2hy + (h + y)^2 \right] dy = \\
= \int_{-h}^{\eta} \{ h^2 + 4hy + 2y^2 \} dy = \\
= h^2(\eta + h) + 2hy^2|_{-h}^{\eta} + \frac{2}{3}y^3|_{-h}^{\eta} = \\
= h^2\eta + 2h\eta^2 + \frac{2}{3}\eta^3 + h^3 - 2h^3 + \frac{2}{3}h^3 = \\
= h^2\eta + 2h\eta^2 + \frac{2}{3}\eta^3 - \frac{1}{3}h^3.
\]

Finally

\[
T = \frac{1}{2} \int \left( \frac{\partial \Psi}{\partial x} \right)^2 (\eta + h) dx + \\
+ \int \left[ h^2\eta \left( 1 + 2 \frac{\eta}{h} + \frac{2}{3} \frac{\eta^3}{h^2} \right) - \frac{1}{6}h^3 \right] \left( \frac{\partial^2 \Psi}{\partial x^2} \right)^2 dx \\
\tag{4.11}
U \simeq \frac{1}{2}g \int \eta^2 dx + \sigma \int \left( \frac{\partial \eta}{\partial x} \right)^2 dx
\]

Using weakly nonlinear approximation \( \eta \ll h \) and \( k\eta \to 0 \) one can get

\[
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\eta + h) \frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{3}h^3 \frac{\partial^4 \Psi}{\partial x^4} \\
\frac{\partial \Psi}{\partial t} + \frac{1}{2} \left( \frac{\partial \Psi}{\partial x} \right)^2 + g\eta = -\sigma \frac{\partial^2 \eta}{\partial x^2} \tag{4.12}
\]

This is a system of type (2.1),(2.2)

In the linear approximation

\[
-i\omega \eta - \hbar k^2 \left( 1 - \frac{1}{3} \hbar^2 k^2 \right) \psi = 0 \\
-i\omega \psi + (g + \sigma k^2) \eta = 0 \\
\omega^2 = g\hbar k^2 \left( 1 - \frac{1}{3} \hbar^2 k^2 \right) \left( 1 + \frac{\sigma}{g} k^2 \right)
\]

17
Appendix to Lecture 4

Hamiltonian description of inhomogeneous fluid.

The most instructive way to prove the theorem on Hamiltonian description of the free surface hydrodynamics is considered in treating this particular case as a particular case of much more general concept — Hamiltonian description of inhomogeneous fluid in a gravitational field. Suppose that a fluid fills a half-infinite domain $\mathbb{R}^2$:

$$b(x, y) < z < \infty$$

The kinetic energy of the fluid is

$$T = \frac{1}{2} \int \nabla v^2 \, d\mathbf{r} \quad (4.13)$$

Potential energy is

$$U = g \int z \rho \, d\mathbf{r} \quad (4.14)$$

This integral might be divergent at $z = \infty$. To avoid divergence we assume that the density at $z = \infty$ tends to a constant value $\rho_c$ and subtract from $U$ and replace

$$U \rightarrow U - U_0 \quad U_0 = \int z \rho_c \, d\mathbf{r}$$
Now Lagrangian of the fluid is

\[ L = T - U \]

However we must take into account that the fluid is incompressible and the fact that inhomogeneity is a passive scalar. Thus we have to accomplish the Lagrangian with two additional terms containing Lagrangian multipliers.

\[ L = T - u + \int \phi \text{div} \mathbf{v} \, dv + \int \lambda (\frac{\partial \phi}{\partial t} + \text{curl} \phi \times \mathbf{b}) \, dv \]  

(4.15)

On this level we consider that \( \mathbf{v}, \phi, \lambda \) are all independent and we have to calculate variational derivative over them separately.

This gives incompressibility condition

\[ \text{div} \mathbf{v} = 0 \]

This comes from \( \frac{\partial L}{\partial u} = 0 \)

The condition \( \frac{\partial L}{\partial \phi} = 0 \) gives equation for transport of inhomogeneity as a passive scalar.
\[
\frac{\partial f}{\partial t} + (\nabla \phi) \cdot \mathbf{v} = 0
\]  \hspace{1cm} (4.17)

Equation \[ \frac{\partial L}{\partial v} = 0 \] leads to an important relation
\hspace{1cm} (4.18)

\[ V = \frac{1}{g} \left( \phi + \lambda \phi \right) \]

It means that potential \( \phi \) can be found from equation (4.18) or from equation (4.19)
\hspace{1cm} (4.19)
\[ \text{div} \frac{1}{g} \phi = \text{div} \frac{1}{g} \phi \]

Let \( S = \int L dt \) and
\[ \text{Requirement} \]
\[ \frac{\partial S}{\partial \phi} = 0 \]

gives evolutionary equation for \( \lambda \)
\[ \frac{\partial \lambda}{\partial t} + \text{div} \lambda V + g \phi - \frac{1}{2} V^2 = 0 \]  \hspace{1cm} (4.20)

Because of conditions (4.18) and (4.19)

only \( \lambda \) and \( \phi \) are really independent variables. Let us consider the total energy
\[ H = T + V \]  \hspace{1cm} (4.21)
and find variational derivatives $\frac{\delta H}{\delta x} > \frac{\delta H}{\delta y}$.

To do this we make one important observation:

In virtue of (4.19), any variation of $\varphi$ and $\lambda$ causes a certain variation of $\Phi$. However, this variation drops out of consideration.

Indeed, it comes to variation of kinetic energy in combination

$$F = \int \sqrt{v^2 \varphi^2} \, dC = -\int \sqrt{v^2 \varphi^2} \, dC = 0$$  \hspace{1cm} (4.22)

Then

$$\frac{\delta F}{\delta \varphi} = \frac{\delta H}{\delta \varphi} = - (\nabla \varphi \Phi)$$

Then equation (4.17) can be rewritten as follows

$$\frac{\delta \varphi}{\delta t} = \frac{\delta H}{\delta \lambda}$$  \hspace{1cm} (4.23)

Calculating $\frac{\delta H}{\delta \varphi}$ we again will not touch $\Phi$.

Now

$$\frac{\delta H}{\delta \varphi} = \varphi \frac{\delta \varphi}{\delta x}$$

Then

$$\int \varphi \frac{\delta \varphi}{\delta x} \, dt = \int \left[ \frac{v}{2} \varphi^2 - \frac{v^2 \varphi^2}{2} \right] \, dt$$
Finally
\[ \frac{\delta P}{\delta \gamma} = - \frac{1}{2} \nabla^2 \gamma + \text{div} \chi \nu \] (4.24)

And equation (4.20) takes form
\[ \frac{\partial \lambda}{\partial t} = - \frac{\delta H}{\delta \gamma} \] (4.25)

Formulas (4.23), (4.25) show that \( \lambda, \gamma \) are canonically conjugated variables.

Equation (4.18) shows that we took into consideration only a special class of fluid motions. Indeed, from (4.18) one obtains

\[ \text{curl } \nu = - \frac{1}{\rho^2} [\nabla \phi, \nabla \nu] - [\nabla \frac{1}{\rho}, \nabla \phi] \]

Hence
\[ (\nabla \phi, \text{curl } \nu) = 0 \] (4.26)

Condition (4.26) shows that vortex lines are directed along surfaces of constant density.

Equations (4.23) (4.25) demonstrate that motion equations realize an extremum of following action

\[ S = \int \frac{1}{2} \rho \nu \text{div} \nu \, dt - H \] (4.27)
Suppose now that the density is piecewise constant

\[ \phi(z, y) = \phi_1 + (\phi_2 - \phi_1) \theta(z - \eta(z, t)) \quad (4.28) \]

here \( \theta = (x_1, y) \) — two-dimensional vector

\[ \eta = \eta(z, t) \] — shape of interface

between domains of constant densities \( \phi_1, \phi_2 \).

Now

\[ \phi_t = (\phi_1 - \phi_2) \eta_t \theta(z - \eta(z, t)) \quad (4.29) \]

\[ S = \int \psi \eta_t \theta z - \eta(z, t) \quad (4.30) \]

\[ \psi = (\phi_1 - \phi_2) \theta z - \eta \]

Value of \( \psi \) on the interface can be found from the condition that velocity is normal to the interface and has no singularity on the interface. It means that potential \( \phi \) has jump

\[ \psi = \frac{\phi_1 - \phi_2}{\phi_1 - \phi_2} \quad (4.31) \]

Finally

\[ \psi = \phi_1 - \phi_2 \]

In the case of surface waves one has to put \( \delta_1 = 1, \delta_2 = 0, \varphi_2 = 0 \) and so on. Now

\[ \nu = \nu' \]

and \( \lambda = \varphi_1 \big|_2 = \lambda \)

as was asserted before.