Lecture 6
Kadomtsev–Petviashvili equation.

In this lecture we derived the universal equation describing long, quasi-one dimensional waves of small amplitude in all versions of the Hydrodynamic wave dispersion. We will study solutions of this system which are close to of simple wave of small amplitude. Thus will assume that the density variation is small

\[ \frac{\delta \rho}{\rho_0} \ll 1 \quad (6.1) \]

\[ \delta \rho = \rho_0 + \delta \rho_0 \]

and expand the simple wave velocity function \( S(\delta) \) in powers of \( \frac{\delta \rho}{\rho_0} \)

\[ S(\delta) = C \left( 1 + \lambda \frac{\delta \rho}{\rho_0} + \ldots \right) \quad (6.2) \]

\( \lambda \) is some dimensionless constant.

In the case of polytropic gas \( \lambda \) can be found from

\[ \lambda = \left. \frac{\delta S}{\delta P} \right|_{p = \rho_0} \quad (6.2) \]

In all realistic models of a continuous medium \( \lambda > 0 \).
In this approximation equation (4.2) reads
\[
\frac{\partial \phi}{\partial t} + c \left( 1 + \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2} \right) \frac{\partial \phi}{\partial x} = 0 \quad (4.3)
\]
This equation must be accomplished by high-order terms. To do this we consider the dispersion relation (3.15) by taking square root and assuming that \( \epsilon^2 k^2 \ll 1 \) one get
\[
\omega = c k_1 \left( 1 + \frac{1}{2} \epsilon^2 k^2 \right) \quad (5.4)
\]
Then we will assume that the wave vector \( \mathbf{k} \) has longitudinal and orthogonal components \( \mathbf{k}_x, \mathbf{k}_z \) and \( |\mathbf{k}_1| < k_2 \)

Then
\[
|\mathbf{k}_1| = \sqrt{k_x^2 + k_2^2} \approx k_x + \frac{1}{2} \frac{k_1^2}{k_x} \quad (6.5)
\]
Finally one can simplify expression (6.4) up to following form
\[
\omega = c \left( k_x + \frac{1}{2} \frac{k_1^2}{k_x} + \frac{1}{2} \epsilon^2 k^3 \right) \quad (6.6)
\]
Now we should modify equation (1.3) such that in the linear approximation it has a solution
\[
\phi = e^{-i\omega(k) t + i\mathbf{k} \cdot \mathbf{x}} \quad (6.4)
\]
There is only one such equation:

\[
\frac{1}{c} \frac{\partial}{\partial t} \delta \phi + \left( 1 + \frac{5\delta\phi}{\delta \phi_0} \right) \frac{\partial}{\partial x} \delta \phi - \frac{1}{2} \varepsilon \frac{\partial^2}{\partial x^2} \delta \phi - \frac{1}{2} \varepsilon \frac{\partial^4}{\partial x^4} \delta \phi = 0
\]

(6.5)

Now we introduce dimensionless variable:

\[
\frac{\delta \rho}{c} = 6u
\]

and rescale time and spatial coordinates:

\[
x \rightarrow L \sqrt{\frac{\varepsilon}{2}} x,
\]

\[
r_\perp \rightarrow L \sqrt{\varepsilon} r_\perp,
\]

\[
t \rightarrow \frac{1}{\sqrt{\varepsilon}} \sqrt{\frac{2}{L}} t,
\]

Then one can go to the moving frame and replace

\[
X \rightarrow x - ct
\]

\[
\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} \right) + \frac{\partial}{\partial \perp} \delta_{\perp} u = 0
\]

This is KP-1 equation.

If \( \varepsilon < 0 \) one has

\[
\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right) + \frac{\partial^2 u}{\partial x^2} = 0 \quad \delta_{\perp} u = 0
\]

(6.7)

This is KP-2 equation.

In absence of dependence on \( \varepsilon \), KP equations reduce to the Korteweg de Vries equations

\[
\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} = 0
\]

(6.8)

\[
\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0
\]

(6.9)

In fact equations (6.8) and (6.9) are equivalent. Equation (6.9) goes to (6.8) after a simple transform \( u \rightarrow -u, \quad t \rightarrow -t \).

Stationary solutions of equation (3.3) obey one of Boussinesq equations.
The same transformation turns KP-1 equation to following

\[ \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right) - \Delta u = 0 \quad (6.10) \]

Equation (6.7), (6.10) are Hamiltonian.

They can be rewritten in following form

\[ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \frac{\partial H}{\partial u} = 0 \quad (6.11) \]

Where for KP-1

\[ H = \int \left( u^3 + \frac{3}{2} u_x^2 + \frac{1}{2} (\nabla^{-1} u)^2 \right) \, dx \, dz_1 \quad (6.12) \]

For KP-2

\[ H = \int \left( u^3 + \frac{3}{2} u_x^2 - \frac{1}{2} (\nabla^{-1} u)^2 \right) \, dx \, dz_1 \quad (6.13) \]

Full three-dimensional KP equations have numerous applications in nonlinear wave dynamics. But they are not integrable systems. Only two-dimensional equations are integrable. They are

\[ \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right) - \Delta \frac{\partial^2 u}{\partial y^2} = 0 \quad (6.14) \]

This is KP-1 equation.
\[
\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} + 6 u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right) + \frac{\partial^2 u}{\partial y^2} = 0
\] (6.15)

This is the KP-2 equation.

In the absence of dependence on \( y \), both equations (6.14) and (6.15) lead to the same KdV equation (6.9). This equation has a solitonic solution (6.16)

\[
u = \frac{k^2}{\cosh^2 k(x-x_0)-4k^2 t} \quad (-\infty < x_0 < \infty)
\]

Here \( k > 0 \) and \( x_0 \) are arbitrary real constants.

An arbitrary real solution of the KdV equation does not depend on \( y \). One can construct solutions depending only on one variable depending on \( t \), which we obtain the KdV equation in a moving frame

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + 6 u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0
\] (6.17)

The sign before \( \frac{\partial^3 u}{\partial x^3} \) in (6.17) is due to a type of KP equation. Equation (6.17) describes following oblique solitons

\[
u = \frac{k^2}{\cosh k(x-x_0)-4k^2 t}.
\]
\[ u = \frac{2k^2}{\cosh^2 k [x - x_0 - ay - (4k^2 + d^2)t]} \]

It is interesting to study stationary waves in the frameworks of KP1 and KP2 equations. To get equations of these waves one has to put

\[ u = u(x + at, y + bt) \]

\[
\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial y^2}
\]

and we end up with the following family of equations

\[
\frac{\partial}{\partial x} \left( a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial x} \left( 6u \frac{\partial u}{\partial x} + \frac{3u^2}{2} \right) \pm \frac{3h}{\sqrt{2}} \to 0
\]

Thereafter we will consider only special cases

Choosing different combinations of signs of constants \(a\) and \(b\) we will obtain four canonical equations known as Boussinesq equations

\(b = 0\) \(a = \pm 1\)
If we linearize these equations and put \( u = q \), we get the following dispersion relation:

\[
\begin{align*}
q^2 &= p^2 + p^4 \\
qu^2 &= p^2 - p^4 \\
qu^2 &= -p^2 + p^4 \\
qu^2 &= -p^2 - p^4
\end{align*}
\]

One can see that only equation (6.20) has linearly stable solution \( u_0 = 0 \). In all other cases, this ground state is unstable. Equations (6.21) and (6.23) are badly ill-posed, while in (6.22) the instability takes place only for long enough waves \( q^2 < 1 \).