Chapter 3

Fourier Series

3.1 Introduction

In solving partial differential equations by the method of separation of variables, we have discovered that important conditions [e.g., the initial condition, \( u(x, 0) = f(x) \)] could be satisfied only if \( f(x) \) could be equated to an infinite linear combination of eigenfunctions of a given boundary value problem. Three specific cases have been investigated. One yielded a series involving sine functions, one yielded a series of cosines only (including a constant term), and the third yielded a series that included all of these previous terms.

We will begin by investigating series with both sines and cosines, because we will show that the others are just special cases of this more general series. For problems with the periodic boundary conditions on the interval \( -L \leq x \leq L \), we asked whether the following infinite series (known as a Fourier series) makes sense:

\[
f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.
\]  

(3.1.1)

Does the infinite series converge? Does it converge to \( f(x) \)? Is the resulting infinite series really a solution of the partial differential equation (and does it also satisfy all the other subsidiary conditions)? Mathematicians tell us that none of these questions have simple answers. Nonetheless, Fourier series usually work quite well (especially in situations where they arise naturally from physical problems). Joseph Fourier developed this type of series in his famous treatise on heat flow in the early 1800s.

The first difficulty that arises is that we claim (3.1.1) will not be valid for all functions \( f(x) \). However, (3.1.1) will hold for some kinds of functions and will need only a small modification for other kinds of functions. In order to communicate various concepts easily, we will discuss only functions \( f(x) \) that are piecewise smooth. A function \( f(x) \) is piecewise smooth (on some interval) if the interval can be broken up into pieces (or sections) such that in each piece the function \( f(x) \) is con-
continuous\(^1\) and its derivative \(df/dx\) is also continuous. The function \(f(x)\) may not be continuous but the only kind of discontinuity allowed is a finite number of jump discontinuities. A function \(f(x)\) has a **jump discontinuity** at a point \(x = x_0\) if the limit from the left \([f(x_0^-)]\) and the limit from the right \([f(x_0^+)]\) both exist (and are unequal), as illustrated in Fig. 3.1.1. An example of a piecewise smooth function is sketched in Fig. 3.1.2. Note that \(f(x)\) has two jump discontinuities at \(x = x_1\) and at \(x = x_3\). Also, \(f(x)\) is continuous for \(x_1 \leq x \leq x_3\) but \(df/dx\) is not continuous for \(x_1 \leq x \leq x_3\). Instead, \(df/dx\) is continuous for \(x_1 \leq x \leq x_2\) and \(x_2 \leq x \leq x_3\). The interval can be broken up into pieces in which both \(f(x)\) and \(df/dx\) are continuous. In this case there are four pieces, \(x \leq x_1\), \(x_1 \leq x \leq x_2\), \(x_2 \leq x \leq x_3\), and \(x \geq x_3\). Almost all functions occurring in practice (and certainly most that we discuss in this book) will be piecewise smooth. Let us briefly give an example of a function that is not piecewise smooth. Consider \(f(x) = x^{1/3}\), as sketched in Fig. 3.1.3. It is not piecewise smooth on any interval that includes \(x = 0\), because \(df/dx = 1/3x^{-2/3}\) is \(\infty\) at \(x = 0\). In other words, any region including \(x = 0\) cannot be broken up into pieces such that \(df/dx\) is continuous.

Each function in the Fourier series is periodic with period \(2L\). Thus, the **Fourier series of** \(f(x)\) **on** the interval \(-L \leq x \leq L\) is periodic with period \(2L\). The function

---

\(^1\)We do not give a definition of a continuous function here. However, one known useful fact is that if a function approaches \(\infty\) at some point, then it is not continuous in any interval including that point.
Figure 3.1.4 Periodic extension of \( f(x) = \frac{3}{2}x \).

\( f(x) \) does not need to be periodic. We need the periodic extension of \( f(x) \). To sketch the periodic extension of \( f(x) \), simply sketch \( f(x) \) for \(-L \leq x \leq L\) and then continually repeat the same pattern with period \( 2L \) by translating the original sketch for \(-L \leq x \leq L\). For example, let us sketch in Fig. 3.1.4 the periodic extension of \( f(x) = \frac{3}{2}x \) [the function \( f(x) = \frac{3}{2}x \) is sketched in dotted lines for \(|x| > L\)]. Note the difference between \( f(x) \) and its periodic extension.

### 3.2 Statement of Convergence Theorem

**Definitions of Fourier coefficients and a Fourier series.** We will be forced to distinguish carefully between a function \( f(x) \) and its Fourier series over the interval \(-L \leq x \leq L\):

\[
\text{Fourier series } = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{3.2.1}
\]

The infinite series may not even converge, and if it converges, it may not converge to \( f(x) \). However, if the series converges, we learned in Chapter 2 how to determine the Fourier coefficients \( a_0, a_n, b_n \) using certain orthogonality integrals, (2.3.32). We will use those results as the definition of the Fourier coefficients:

\[
\begin{align*}
a_0 & = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx \\
a_n & = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx \\
b_n & = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx.
\end{align*} \tag{3.2.2}
\]
The Fourier series of \( f(x) \) over the interval \(-L \leq x \leq L\) is defined to be the infinite series (3.2.1), where the Fourier coefficients are given by (3.2.2). We immediately note that a Fourier series does not exist unless for example \( a_0 \) exists [i.e., unless \( \int_{-L}^{L} f(x) \, dx < \infty \)]. This eliminates certain functions from our consideration. For example, we do not ask what is the Fourier series of \( f(x) = 1/x^2 \).

Even in situations in which \( \int_{-L}^{L} f(x) \, dx \) exists, the infinite series may not converge; furthermore, if it converges, it may not converge to \( f(x) \). We use the notation

\[
f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},
\]

(3.2.3)

where \( \sim \) means that \( f(x) \) is on the left-hand side and the Fourier series of \( f(x) \) (on the interval \(-L \leq x \leq L\)) is on the right-hand side (even if the series diverges), but the two functions may be completely different. The symbol \( \sim \) is read as "has the Fourier series (on a given interval)."

**Convergence theorem for Fourier series.** At first we state a theorem summarizing certain properties of Fourier series:

If \( f(x) \) is piecewise smooth on the interval \(-L \leq x \leq L\), then the Fourier series of \( f(x) \) converges

1. to the periodic extension of \( f(x) \), where the periodic extension is continuous;
2. to the average of the two limits, usually
\[
\frac{1}{2} [f(x+) + f(x-)],
\]

where the periodic extension has a jump discontinuity.

We refer to this as Fourier's theorem. It is proved in many of the references listed in the Bibliography.

Mathematically, if \( f(x) \) is piecewise smooth, then for \(-L < x < L\) (excluding the endpoints),

\[
\frac{f(x+) + f(x-)}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},
\]

(3.2.4)

where the Fourier coefficients are given by (3.2.2). At points where \( f(x) \) is continuous, \( f(x+) = f(x-) \) and hence (3.2.4) implies that for \(-L < x < L\),

\[
f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.
\]
The Fourier series actually converges to \( f(x) \) at points between \(-L\) and \(+L\), where \( f(x) \) is continuous. At the endpoints, \( x = L \) or \( x = -L \), the infinite series converges to the average of the two values of the periodic extension. Outside the range \(-L \leq x \leq L\), the Fourier series converges to a value easily determined using the known periodicity (with period \(2L\)) of the Fourier series.

**Sketching Fourier series.** Now we are ready to apply Fourier's theorem. To sketch the Fourier series of \( f(x) \) (on the interval \(-L \leq x \leq L\)), we

1. Sketch \( f(x) \) (preferably for \(-L \leq x \leq L\) only).
2. Sketch the periodic extension of \( f(x) \).

According to Fourier's theorem, the Fourier series converges (here converge means "equals") to the periodic extension, where the periodic extension is continuous (which will be almost everywhere). However, at points of jump discontinuity of the periodic extension, the Fourier series converges to the average. Therefore, there is a third step:

3. Mark an "\( \times \)" at the average of the two values at any jump discontinuity of the periodic extension.

**Example.** Consider

\[
f(x) = \begin{cases} 
\frac{L}{2} & x < \frac{L}{2} \\
1 & x > \frac{L}{2}.
\end{cases}
\]  

(3.2.5)

We would like to determine the Fourier series of \( f(x) \) on \(-L \leq x \leq L\). We begin by sketching \( f(x) \) for all \( x \) in Fig. 3.2.1 (although we only need the sketch for \(-L \leq x \leq L\).) Note that \( f(x) \) is piecewise smooth, so we can apply Fourier's theorem. The periodic extension of \( f(x) \) is sketched in Fig. 3.2.2. Often the understanding of the process is made clearer by sketching at least three full periods, \(-3L \leq x \leq 3L\), even though in the applications to partial differential equations only the interval \(-L \leq x \leq L\) is absolutely needed. The Fourier series of \( f(x) \) equals the periodic extension of \( f(x) \), wherever the periodic extension is continuous (i.e., at all \( x \) except the points of jump discontinuity, which are \( x = L/2, L, L/2 + 2L, -L, L/2 - 2L, \) etc.). According to Fourier's theorem, at these points of jump discontinuity, the Fourier series of \( f(x) \) must converge to the average. These points should be marked, perhaps with an \( \times \), as in Fig. 3.2.2. At \( x = L/2 \) and \( x = L \) (as well as \( x = L/2 \pm 2nL \) and \( x = L \pm 2nL \)), the Fourier series converges to the average, \( \frac{1}{2} \). In summary, for
\[ f(x) \]

\[ \frac{L}{2} \]

**Figure 3.2.1** Sketch of \( f(x) \)

\[\begin{array}{c|c|c|c|c|c}
-3L & -L & 0 & L/2 & L & 3L \\
\hline
x & x & x & x & x \\
\end{array}\]

**Figure 3.2.2** Fourier series of \( f(x) \).

In this example,

\[
a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} = \begin{cases} 
\frac{1}{2} & x = -L \\
0 & -L < x < L/2 \\
\frac{1}{2} & x = L/2 \\
1 & L/2 < x < L \\
\frac{1}{2} & x = L.
\end{cases}
\]

Fourier series can converge to rather strange functions, but they are not so different from the original function.

**Fourier coefficients.** For a given \( f(x) \), it is *not* necessary to calculate the Fourier coefficients in order to sketch the Fourier series of \( f(x) \). However, it is important to know how to calculate the Fourier coefficients, given by (3.2.2). The calculation of Fourier coefficients can be an algebraically involved process. Sometimes it is an exercise in the method of integration by parts. Often, calculations can be simplified by judiciously using integral tables or computer algebra systems. In any event, we can always use a computer to approximate the coefficients numerically. As an overly simple example but one that illustrates some important points, consider \( f(x) \) given by (3.2.5). From (3.2.2), the coefficients are

\[
a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx = \frac{1}{2L} \int_{L/2}^{L} dx = \frac{1}{4} \quad \text{(3.2.6)}
\]
3.2. Convergence Theorem.

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx = \frac{1}{L} \int_{L/2}^{L} \cos \frac{n\pi x}{L} \, dx = \frac{1}{n\pi} \sin \frac{n\pi x}{L} \Bigg|_{L/2}^{L} \]

\[ = \frac{1}{n\pi} \left( \sin n\pi - \sin \frac{n\pi}{2} \right) \] (3.2.7)

\[ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx = \frac{1}{L} \int_{L/2}^{L} \sin \frac{n\pi x}{L} \, dx = -\frac{1}{n\pi} \cos \frac{n\pi x}{L} \Bigg|_{L/2}^{L} \]

\[ = \frac{1}{n\pi} \left( \cos \frac{n\pi}{2} - \cos n\pi \right) \] (3.2.8)

We omit simplifications that arise by noting that \( \sin n\pi = 0 \), \( \cos n\pi = (-1)^n \), and so on.

EXERCISES 3.2

3.2.1. For the following functions, sketch the Fourier series of \( f(x) \) (on the interval \( -L \leq x \leq L \)). Compare \( f(x) \) to its Fourier series:

(a) \( f(x) = 1 \)

(c) \( f(x) = 1 + x \)

(e) \( f(x) = \begin{cases} x & x < 0 \\ 2x & x > 0 \end{cases} \)

(g) \( f(x) = \begin{cases} x & x < L/2 \\ 0 & x > L/2 \end{cases} \)

(b) \( f(x) = x^2 \)

(d) \( f(x) = e^x \)

(f) \( f(x) = \begin{cases} 0 & x < 0 \\ 1 + x & x > 0 \end{cases} \)

3.2.2. For the following functions, sketch the Fourier series of \( f(x) \) (on the interval \( -L \leq x \leq L \)) and determine the Fourier coefficients:

* (a) \( f(x) = x \)

* (c) \( f(x) = \sin \frac{\pi x}{L} \)

* (e) \( f(x) = \begin{cases} 1 & |x| < L/2 \\ 0 & |x| > L/2 \end{cases} \)

* (g) \( f(x) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases} \)

(b) \( f(x) = e^{-x} \)

(d) \( f(x) = \begin{cases} 0 & x < 0 \\ x & x > 0 \end{cases} \)

(f) \( f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \)
3.2.3. Show that the Fourier series operation is linear: that is, show that the Fourier series of \( c_1 f(x) + c_2 g(x) \) is the sum of \( c_1 \) times the Fourier series of \( f(x) \) and \( c_2 \) times the Fourier series of \( g(x) \).

3.2.4. Suppose that \( f(x) \) is piecewise smooth. What value does the Fourier series of \( f(x) \) converge to at the endpoint \( x = -L \) at \( x = L \)?

3.3 Fourier Cosine and Sine Series

In this section we show that the series of sines only (and the series of cosines only) are special cases of a Fourier series.

3.3.1 Fourier Sine Series

Odd functions. An odd function is a function with the property \( f(-x) = -f(x) \). The sketch of an odd function for \( x < 0 \) will be minus the mirror image of \( f(x) \) for \( x > 0 \), as illustrated in Fig. 3.3.1. Examples of odd functions are \( f(x) = x^3 \) (in fact, any odd power) and \( f(x) = \sin 4x \). The integral of an odd function over a symmetric interval is zero (any contribution from \( x > 0 \) will be canceled by a contribution from \( x < 0 \)).

![Figure 3.3.1 An odd function.](image)

**Fourier series of odd functions.** Let us calculate the Fourier coefficients of an odd function:

\[
\begin{align*}
a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) \, dx = 0 \\
ap &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx = 0.
\end{align*}
\]

Both are zero because the integrand, \( f(x) \cos n\pi x / L \), is odd [being the product of an even function \( \cos n\pi x / L \) and an odd function \( f(x) \)]. Since \( a_n = 0 \), all the cosine functions (which are even) will not appear in the Fourier series of an odd function. The Fourier series of an odd function is an infinite series of odd functions (sines):

\[
f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},
\]  

(3.3.1)
if \( f(x) \) is odd. In this case formulas for the Fourier coefficients \( b_n \) may be simplified:

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx, \tag{3.3.2}
\]

since the integral of an even function over the symmetric interval \(-L\) to \(+L\) is twice the integral from 0 to \( L \). For odd functions information about \( f(x) \) is needed only for \( 0 \leq x \leq L \).

**Fourier sine series.** However, only occasionally are we given an odd function and asked to compute its Fourier series. Instead, frequently series of only sines arise in the context of separation of variables. Recall that the temperature in a one-dimensional rod \( 0 < x < L \) with zero temperature ends \( u(0, t) = u(L, t) = 0 \) satisfies

\[
u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\left(\frac{n\pi}{L}\right)^2 kt}, \tag{3.3.3}
\]

where the initial condition \( u(x, 0) = f(x) \) is satisfied if

\[
f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}. \tag{3.3.4}
\]

\( f(x) \) must be represented as a series of sines; (3.3.4) appears in the same form as (3.3.1). However, there is a significant difference. In (3.3.1) \( f(x) \) is given as an odd function and defined for \(-L \leq x \leq L\). In (3.3.4) \( f(x) \) is only defined for \( 0 \leq x \leq L \) (it is just the initial temperature distribution); \( f(x) \) is certainly not necessarily odd. If \( f(x) \) is only given for \( 0 \leq x \leq L \), then it can be extended as an odd function; see Fig. 3.3.2, called the **odd extension of \( f(x) \)**. The odd extension of \( f(x) \) is defined for \(-L \leq x \leq L\). Fourier’s theorem will apply [if the odd extension of \( f(x) \) is piecewise smooth, which just requires that \( f(x) \) is piecewise smooth for \( 0 \leq x \leq L \)]. Moreover, since the odd extension of \( f(x) \) is certainly odd, its Fourier series only involves sines:

\[
\text{the odd extension of } f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad -L \leq x \leq L,
\]

where \( B_n \) are given by (3.3.2). However, we are only interested in what happens between \( x = 0 \) and \( x = L \). In that region \( f(x) \) is identical to its odd extension:

\[
f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \quad 0 \leq x \leq L, \tag{3.3.5}
\]

where

\[
B_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx. \tag{3.3.6}
\]
We call this the Fourier sine series of \( f(x) \) (on the interval \( 0 \leq x \leq L \)). This series (3.3.5) is nothing but an example of a Fourier series. As such, we can simply apply Fourier’s theorem; just remember that \( f(x) \) is only defined for \( 0 \leq x \leq L \). We may think of \( f(x) \) as being odd (although it is not necessarily) by extending \( f(x) \) as an odd function. Formula (3.3.6) is very important but does not need to be memorized. It can be derived from the formulas for a Fourier series simply by assuming that \( f(x) \) is odd. [It is more accurate to say that we consider the odd extension of \( f(x) \)].. Formula (3.3.6) is a factor of 2 larger than the Fourier series coefficients since the integrand is even. In (3.3.6) the integrals are only from \( x = 0 \) to \( x = L \).

According to Fourier’s theorem, sketching the Fourier sine series of \( f(x) \) is easy:

1. Sketch \( f(x) \)(for \( 0 < x < L \)).
2. Sketch the odd extension of \( f(x) \).
3. Extend as a periodic function (with period \( 2L \)).
4. Mark an \( \times \) at the average at points where the odd periodic extension of \( f(x) \) has a jump discontinuity.

**Example.** As an example, we show how to sketch the Fourier sine series of \( f(x) = 100 \). We consider \( f(x) = 100 \) only for \( 0 \leq x \leq L \). We begin by sketching in Fig. 3.3.3 its odd extension. The Fourier sine series of \( f(x) \) equals the Fourier series of the odd extension of \( f(x) \). In Fig. 3.3.4 we repeat periodically the odd extension (with period \( 2L \)). At points of discontinuity, the average is marked with an \( \times \). According to Fourier’s theorem (as illustrated in Fig. 3.3.4), the Fourier sine series of 100 actually equals 100 for \( 0 < x < L \), but the infinite series does not equal 100 at \( x = 0 \) and \( x = L \):

\[
100 = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad 0 < x < L. \tag{3.3.7}
\]

At \( x = 0 \), Fig. 3.3.4 shows that the Fourier sine series converges to 0, because at \( x = 0 \) the odd property of the sine series yields the average of 100 and \(-100\), which
is 0. For similar reasons, the Fourier sine series also converges to 0 at \( x = L \). These observations agree with the result of substituting \( x = 0 \) (and \( x = L \)) into the infinite series of sines. The Fourier coefficients are determined from (3.3.6) as before [see (2.3.42)]:

\[
B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx = \frac{200}{L} \int_0^L \sin \frac{n\pi x}{L} \, dx = \begin{cases} 0 & n \text{ even} \\ \frac{400}{n\pi} & n \text{ odd} \end{cases} \quad (3.3.8)
\]

**Physical example.** One of the simplest examples is the Fourier sine series of a constant. This problem arose in trying to solve the one-dimensional heat equation with zero boundary conditions and constant initial temperature, 100°:

**PDE:** \( \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \ t > 0 \)

**BC1:** \( u(0, t) = 0 \)

**BC2:** \( u(L, t) = 0 \)

**IC:** \( u(x, 0) = f(x) = 100°. \)

We recall from Sec. 2.3 that the method of separation of variables implied that

\[
u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}.
\]

The initial conditions are satisfied if

\[
100 = f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad 0 < x < L.
\]
This may be interpreted as the Fourier sine series of \( f(x) = 100 \) [see (3.3.8)]. Equivalently, \( B_n \) may be determined from the orthogonality of \( \sin n\pi x/L \) [see (2.3.42)].

Mathematically, the Fourier series of the initial condition has a rather bizarre behavior at \( x = 0 \) (and at \( x = L \)). In fact, for this problem, the physical situation is not very well defined at \( x = 0 \) (at \( t = 0 \)). This might be illustrated in a space-time diagram, Fig. 3.3.5. We note that Fig. 3.3.5 shows that the domain of our problem is \( t \geq 0 \) and \( 0 \leq x \leq L \). However, there is a conflict that occurs at \( x = 0, \ t = 0 \) between the initial condition and the boundary condition. The initial condition \( (t = 0) \) prescribes the temperature to be 100° even as \( x \to 0 \), whereas the boundary condition \( (x = 0) \) prescribes the temperature to be 0° even as \( t \to 0 \). Thus, the physical problem has a discontinuity at \( x = 0, \ t = 0 \). In the actual physical world, the temperature cannot be discontinuous. We introduced a discontinuity into our mathematical model by "instantaneously" transporting (at \( t = 0 \)) the rod from a 100° bath to a 0° bath at \( x = 0 \). It actually takes a finite time, and the temperature would be continuous. Nevertheless, the transition from 0° to 100° would occur over an exceedingly small distance and time. We introduce the temperature discontinuity to approximate the more complicated real physical situation. Fourier's theorem thus illustrates how the physical discontinuity at \( x = 0 \) (initially, at \( t = 0 \)) is reproduced mathematically. The Fourier sine series of 100° (which represents the physical solution at \( t = 0 \)) has the nice property that it equals 100° for all \( x \) inside the rod, \( 0 < x < L \) (thus satisfying the initial condition there), but it equals 0° at the boundaries, \( x = 0 \) and \( x = L \) (thus also satisfying the boundary conditions). The Fourier sine series of 100° is a strange mathematical function, but so is the physical approximation for which it is needed.

**Fourier series computations and the Gibbs phenomenon.** Let us gain some confidence in the validity of Fourier series. The Fourier sine series of \( f(x) = 100 \) states that

\[
100 = \frac{400}{\pi} \left( \frac{\sin \pi x/L}{1} + \frac{\sin 3\pi x/L}{3} + \frac{\sin 5\pi x/L}{5} + \cdots \right). \tag{3.3.10}
\]

Do we believe (3.3.10)? Certainly, it is not valid at \( x = 0 \) (as well as the other boundary \( x = L \)), since at \( x = 0 \) every term in the infinite series is zero (they
cannot add to 100). However, the theory of Fourier series claims that (3.3.10) is valid everywhere except the two ends. For example, we claim it is valid at \( x = L/2 \). Substituting \( x = L/2 \) into (3.3.10) shows that

\[
100 = \frac{400}{\pi} \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots \right) \quad \text{or} \quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots.
\]

At first this may seem strange. However, it is Euler's formula for \( \pi \). It can be used to compute \( \pi \) (although very inefficiently); it can also be shown to be true without relying on the theory of infinite trigonometric series (see Exercise 3.3.17). The validity of (3.3.10) for other values of \( x \), \( 0 < x < L \), may also surprise you. We will sketch the left- and right-hand sides of (3.3.10), hopefully convincing you of their equality. We will sketch the r.h.s. by adding up the contribution of each term of the series. Of course, we cannot add up the required infinite number of terms; we will settle for a finite number of terms. In fact, we will sketch the sum of the first few terms to see how the series approaches the constant 100 as the number of terms increases. It is helpful to know that \( 400/\pi = 127.32395 \ldots \) (although for rough sketching 125 or 130 will do). The first term \( (400/\pi) \sin \pi x/L \) by itself is the basic first rise and fall of a sine function; it is not a good approximation to the constant 100, as illustrated in Fig. 3.3.6. On the other hand, for just one term in an infinite series it is not such a bad approximation. The next term to be added is \( (400/3\pi) \sin 3\pi x/L \). This is a sinusoidal oscillation, with one-third the amplitude and one-third the period of the first term. It is positive near \( x = 0 \) and \( x = L \), where the approximation needs to be increased, and it is negative near \( x = L/2 \), where the approximation needs to be decreased. It is sketched in dashed lines and then added to the first term in Fig. 3.3.7. Note that the sum of the two nonzero terms already seems to be a considerable improvement over the first term. Computer plots of some partial sums are given in Fig. 3.3.8.

Actually, a lot can be learned from Fig. 3.3.8. Perhaps now it does seem reasonable that the infinite series converges to 100 for \( 0 < x < L \). The worst places (where the finite series differs most from 100) are getting closer and closer to \( x = 0 \) and \( x = L \) as the number of terms increases. For a finite number of terms in the series, the solution starts from zero at \( x = 0 \) and shoots up beyond 100, what we call the primary overshoot. It is interesting to note that Fig. 3.3.8 illustrates the overshoot vividly. We can even extrapolate to guess what happens for 1000 terms. The series should become more and more accurate as the number of terms increases. We might expect the overshoot to vanish as \( n \to \infty \), but put a straight edge on the points of maximum overshoot. It just does not seem to approach 100. Instead, it is far away from that, closer to 118. This overshoot is an example of the Gibbs phenomenon. In general (for large \( n \)), there is an overshoot (and corresponding undershoot) of approximately 9% of the jump discontinuity. In this case (see Fig. 3.3.4), the Fourier sine series of \( f(x) = 100 \) jumps from \(-100\) to \(+100\) at \( x = 0 \). Thus, the finite series will overshoot by about 9% of 200, or approximately 18. The Gibbs phenomenon occurs only when a finite series of eigenfunctions approximates a discontinuous function.
Figure 3.3.6 First term of Fourier sine series of $f(x) = 100$.

Figure 3.3.7 First two nonzero terms of Fourier sine series of $f(x) = 100$. 
Figure 3.3.8 Various partial sums of Fourier sine series of $f(x) = 100$. Using 51 terms (including $n = 51$), the finite series is a good approximation to $f(x) = 100$ away from the endpoints. Near the endpoints (where there is a jump discontinuity of 200), there is a 9% overshoot (Gibbs phenomenon).
Further example of a Fourier sine series. We consider the Fourier sine series of \( f(x) = x \). \( f(x) = x \) is sketched on the interval \( 0 \leq x \leq L \) in Fig. 3.3.9a. The odd-periodic extension of \( f(x) \) is sketched in Fig. 3.3.9b. The jump discontinuity of the odd-periodic extension at \( x = (2n - 1)L \) shows that, for example, the Fourier sine series of \( f(x) = x \) converges to zero at \( x = L \). While \( f(L) \neq 0 \) we note that the Fourier sine series of \( f(x) = x \) actually equals \( x \) for \( -L < x < L \),

\[
x = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad -L < x < L.
\]  

(3.3.11)

The Fourier coefficients are determined from (3.3.6):

\[
B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx = \frac{2}{L} \int_0^L x \sin \frac{n\pi x}{L} \, dx = \frac{2L}{n\pi} (-1)^{n+1},
\]  

(3.3.12)

where the integral can be evaluated by integration by parts (or by a table).

![Figure 3.3.9](image)

(a) \( f(x) = x \) and (b) its Fourier sine series.

Example. We now consider the Fourier sine series of \( f(x) = \cos \pi x/L \). This may seem to ask for a sine series expansion of an even function, but in applications often the function is only given from \( 0 \leq x \leq L \) and must be expanded in a series of sines due to the boundary conditions. \( \cos \pi x/L \) is sketched in Fig. 3.3.10a. It is an even function, but its odd extension is sketched in Fig. 3.3.10b. The Fourier sine series of \( f(x) \) equals the Fourier series of the odd extension of \( f(x) \). Thus, we repeat the sketch in Fig. 3.3.10b periodically (see Fig. 3.3.11), placing an \( \times \) at the average of the two values at the jump discontinuities. The Fourier sine series representation of \( \cos \pi x/L \) is

\[
\cos \frac{\pi x}{L} \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad 0 \leq x \leq L,
\]  

where with some effort we obtain

\[
B_n = \frac{2}{L} \int_0^L \cos \frac{\pi x}{L} \sin \frac{n\pi x}{L} \, dx = \begin{cases} 
0 & n \text{ odd} \\
\frac{4n}{\pi(n^2-1)} & n \text{ even}
\end{cases}
\]  

(3.3.13)
According to Fig. 3.3.11 (based on Fourier's theorem), equality holds for $0 < x < L$, but not at $x = 0$ and not at $x = L$:

$$\cos \frac{\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad 0 < x < L. $$

At $x = 0$ and at $x = L$ the infinite series must converge to 0, since all terms in the series are zero there. Figure 3.3.11 agrees with this. You may be a bit puzzled by an aspect of this problem. You may have recalled that $\sin n\pi x/L$ is orthogonal to $\cos m\pi x/L$, and thus expected all the $B_n$ in (3.3.12) to be zero. However, $B_n \neq 0$. The subtle point is that you should remember that $\cos m\pi x/L$ and $\sin n\pi x/L$ are orthogonal on the interval $-L \leq x \leq L$, $\int_{-L}^{L} \cos m\pi x/L \sin n\pi x/L \, dx = 0$; they are not orthogonal on $0 \leq x \leq L$. 
3.3.2 Fourier Cosine Series

Even functions. Similar ideas are valid for even functions, in which \( f(-x) = f(x) \). Let us develop the basic results. The sine coefficients of a Fourier series will be zero for an even function,

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx = 0,
\]

since \( f(x) \) is even. The Fourier series of an even function is a representation of \( f(x) \) involving an infinite sum of only even functions (cosines):

\[
f(x) \sim \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L}, \tag{3.3.14}
\]

if \( f(x) \) is even. The coefficients of the cosines may be evaluated using information about \( f(x) \) only between \( x = 0 \) and \( x = L \), since

\[
a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx = \frac{1}{L} \int_{0}^{L} f(x) \, dx \tag{3.3.15}
\]

\[
(n \geq 1) \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} \, dx, \tag{3.3.16}
\]

using the fact that for \( f(x) \) even, \( f(x) \cos n\pi x/L \) is even.

Often, \( f(x) \) is not given as an even function. Instead, in trying to represent an arbitrary function \( f(x) \) using an infinite series of \( \cos n\pi x/L \), the eigenfunctions of the boundary value problem \( d^2\phi/dx^2 = -\lambda \phi \) with \( d\phi/dx (0) = 0 \) and \( d\phi/dx (L) = 0 \), we wanted

\[
f(x) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}, \tag{3.3.17}
\]

only for \( 0 < x < L \). We had previously determined the coefficients \( A_n \) to be the same as given by (3.3.15) and (3.3.16), but our reason was because of the orthogonality of \( \cos n\pi x/L \). To relate (3.3.17) to a Fourier series, we simply introduce the even extension of \( f(x) \), an example being illustrated in Fig. 3.3.12. If \( f(x) \) is piecewise smooth for \( 0 \leq x \leq L \), then its even extension will also be piecewise smooth, and hence Fourier's theorem can be applied to the even extension of \( f(x) \). Since the even extension of \( f(x) \) is an even function, the Fourier series of the even extension of \( f(x) \) will have only cosines:

\[
\text{even extension of } f(x) \sim \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad -L \leq x \leq L,
\]

where \( a_n \) is given by (3.3.15) and (3.3.16). In the region of interest, \( 0 \leq x \leq L \), \( f(x) \) is identical to the even extension. The resulting series in that region is called
the Fourier cosine series of \( f(x) \) (on the interval \( 0 \leq x \leq L \)):

\[
f(x) \sim \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}, \quad 0 \leq x \leq L
\]  

(3.3.18)

\[
A_0 = \frac{1}{L} \int_0^L f(x) \, dx
\]  

(3.3.19)

\[
A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx.
\]  

(3.3.20)

The Fourier cosine series of \( f(x) \) is exactly the Fourier series of the even extension of \( f(x) \). Since we can apply Fourier's theorem, we have an algorithm to sketch the Fourier cosine series of \( f(x) \):

1. Sketch \( f(x) \) (for \( 0 < x < L \)).
2. Sketch the even extension of \( f(x) \).
3. Extend as a periodic function (with period \( 2L \)).
4. Mark \( \times \) at points of discontinuity at the average.

**Example.** We consider the Fourier cosine series of \( f(x) = x \). \( f(x) \) is sketched in Fig. 3.3.13a [note that \( f(x) \) is odd!!]. We consider \( f(x) \) only from \( x = 0 \) to \( x = L \) and then extend it in Fig. 3.3.13b as an even function. Next, we sketch the Fourier series of the even extension, by periodically extending the even extension (see Fig. 3.3.14). Note that between \( x = 0 \) and \( x = L \) the Fourier cosine series has
no jump discontinuities. The Fourier cosine series of \( f(x) = x \) actually equals \( x \), so that

\[
x = \sum_{n=0}^{\infty} A_n \cos \frac{n \pi x}{L}, \quad 0 \leq x \leq L.
\]  \hspace{1cm} (3.3.21)

The coefficients are given by the following integrals:

\[
A_0 = \frac{1}{L} \int_0^L x \, dx = \frac{1}{L} \frac{1}{2} x^2 \bigg|_0^L = \frac{L}{2} \hspace{1cm} (3.3.22)
\]

\[
A_n = \frac{2}{L} \int_0^L x \cos \frac{n \pi x}{L} \, dx = \frac{2L}{(n \pi)^2} (\cos n \pi - 1). \hspace{1cm} (3.3.23)
\]

The latter integral can be evaluated by integration by parts, tables, or a symbolic computation program. We omit the details.

### 3.3.3 Representing \( f(x) \) by Both a Sine and Cosine Series

It may be apparent that any function \( f(x) \) (which is piecewise smooth) may be represented both as a Fourier sine series and as a Fourier cosine series. The one
you would use is dictated by the boundary conditions (if the problem arose in the context of a solution to a partial differential equation using the method of separation of variables). It is also possible to use a Fourier series (including both sines and cosines). As an example, we consider the sketches of the Fourier, Fourier sine, and Fourier cosine series of

\[ f(x) = \begin{cases} 
-\frac{L}{2} \sin \frac{\pi x}{L} & x < 0 \\
x & 0 < x < \frac{L}{2} \\
L - x & x > \frac{L}{2}.
\end{cases} \]

The graph of \( f(x) \) is sketched for \(-L < x < L\) in Fig. 3.3.15. The Fourier series of \( f(x) \) is sketched by repeating this pattern with period \( 2L \). On the other hand, for the Fourier sine (cosine) series, first sketch the odd (even) extension of the function \( f(x) \) before repeating the pattern. These three are sketched in Fig. 3.3.16. Note that for \(-L \leq x \leq L\) only the Fourier series of \( f(x) \) actually equals \( f(x) \). However, for all three cases the series equals \( f(x) \) over the region \( 0 \leq x \leq L \).

![Graph of f(x) for -L < x < L](image)

**Figure 3.3.15** The graph of \( f(x) \) for \(-L < x < L\).

### 3.3.4 Even and Odd Parts

Let us consider the Fourier series of a function \( f(x) \) that is not necessarily even or odd:

\[
f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},
\]

(3.3.24)

where

\[
a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx
\]

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx.
\]
Figure 3.3.16 (a) Fourier series of \( f(x) \); (b) Fourier sine series of \( f(x) \); (c) Fourier cosine series of \( f(x) \).

It is interesting to see that a Fourier series is the sum of a series of cosines and a series of sines. For example, \( \sum_{n=1}^{\infty} b_n \sin n\pi x/L \) is not, in general, the Fourier sine series of \( f(x) \), because the coefficients, \( b_n = 1/L \int_{-L}^{L} f(x) \sin n\pi x/L \ dx \), are not, in general, the same as the coefficients of a Fourier sine series \( 2/L \int_{0}^{L} f(x) \sin n\pi x/L \ dx \). This series of sines by itself ought to be the Fourier sine series of some function; let us determine this function.

Equation (3.3.24) shows that \( f(x) \) is represented as a sum of an even function (for the series of cosines must be an even function) and an odd function (similarly, the sine series must be odd). This is a general property of functions, since for any function it is rather obvious that

\[
f(x) = \frac{1}{2} \left[ f(x) + f(-x) \right] + \frac{1}{2} \left[ f(x) - f(-x) \right].
\]

(3.3.25)

Note that the first bracketed term is an even function; we call it the **even part** of \( f(x) \). The second bracketed term is an odd function, called the **odd part** of \( f(x) \):

\[
f_e(x) \equiv \frac{1}{2} \left[ f(x) + f(-x) \right] \quad \text{and} \quad f_o(x) \equiv \frac{1}{2} \left[ f(x) - f(-x) \right].
\]

(3.3.26)

In this way, any function is written as the sum of an odd function (the odd part) and an even function (the even part). For example, if \( f(x) = 1/(1 + x) \),

\[
\frac{1}{1 + x} = \frac{1}{2} \left[ \frac{1}{1 + x} + \frac{1}{1 - x} \right] + \frac{1}{2} \left[ \frac{1}{1 + x} - \frac{1}{1 - x} \right] = \frac{1}{1 + x^2} - \frac{x}{1 - x^2}.
\]

This is the sum of an even function, \( 1/(1 - x^2) \), and an odd function, \(-x/(1 - x^2)\). Consequently, the Fourier series of \( f(x) \) equals the Fourier series of \( f_e(x) \) [which is
a cosine series since $f_e(x)$ is even] plus the Fourier series of $f_o(x)$ (which is a sine series since $f_o(x)$ is odd). This shows that the series of sines (cosines) that appears in (3.3.17) is the Fourier sine (cosine) series of $f_o(x)(f_e(x))$. We summarize our result with the statement:

\[
\text{The Fourier series of } f(x) \text{ equals the Fourier sine series of } f_o(x) \text{ plus the Fourier cosine series of } f_e(x), \text{ where } f_o(x) = \frac{1}{2} [f(x) + f(-x)], \text{ and } f_e(x) = \frac{1}{2} [f(x) - f(-x)].
\]

Please do not confuse this result with even and odd extensions. For example, the even part of $f(x) = \frac{1}{2} [f(x) + f(-x)]$, while the

\[
\text{even extension of } f(x) = \begin{cases} 
  f(x), & x > 0 \\
  f(-x), & x < 0.
\end{cases}
\]

### 3.3.5 Continuous Fourier Series

The convergence theorem for Fourier series shows that the Fourier series of $f(x)$ may be a different function than $f(x)$. Nevertheless, over the interval of interest, they are the same except at those few points where the periodic extension of $f(x)$ has a jump discontinuity. Sine (cosine) series are analyzed in the same way, where instead the odd (even) periodic extension must be considered. In addition to points of jump discontinuity of $f(x)$ itself, the various extensions of $f(x)$ may introduce a jump discontinuity. From the examples in the preceding section, we observe that sometimes the resulting series does not have any jump discontinuities. In these cases the Fourier series of $f(x)$ will actually equal $f(x)$ in the range of interest. Also, the Fourier series itself will be a continuous function.

It is worthwhile to summarize the conditions under which a Fourier series is continuous:

For piecewise smooth $f(x)$, the Fourier series of $f(x)$ is continuous and converges to $f(x)$ for $-L \leq x \leq L$ if and only if $f(x)$ is continuous and $f(-L) = f(L)$.

It is necessary for $f(x)$ to be continuous; otherwise, there will be a jump discontinuity [and the Fourier series of $f(x)$ will converge to the average]. In Fig. 3.3.17 we illustrate the significance of the condition $f(-L) = f(L)$. We illustrate two continuous functions, only one of which satisfies $f(-L) = f(L)$. The condition $f(-L) = f(L)$ insists that the repeated pattern (with period $2L$) will be continuous at the endpoints. The preceding boxed statement is a fundamental result for all Fourier series. It explains the following similar theorems for Fourier sine and cosine series.

Consider the Fourier cosine series of $f(x)$ [$f(x)$ has been extended as an even function]. If $f(x)$ is continuous, is the Fourier cosine series continuous? An example
that is continuous for $0 \leq x \leq L$ is sketched in Fig. 3.3.18. First we extend $f(x)$ evenly and then periodically. It is easily seen that

> For piecewise smooth $f(x)$, the Fourier cosine series of $f(x)$ is continuous and converges to $f(x)$ for $0 \leq x \leq L$ if and only if $f(x)$ is continuous.

We note that no additional conditions on $f(x)$ are necessary for the cosine series to be continuous (besides $f(x)$ being continuous). One reason for this result is that if $f(x)$ is continuous for $0 \leq x \leq L$, then the even extension will be continuous for
3.3. Cosine and Sine Series

\[ -3L < x < L \] Also note that the even extension is the same at \( \pm L \). Thus, the periodic extension will automatically be continuous at the endpoints.

Compare this result to what happens for a Fourier sine series. Four examples are considered in Fig. 3.3.19, all continuous functions for \( 0 \leq x \leq L \). From the first three figures, we see that it is possible for the Fourier sine series of a continuous function to be discontinuous. It is seen that

\[ \text{For piecewise smooth functions } f(x), \text{ the Fourier sine series of } f(x) \]
\[ \text{is continuous and converges to } f(x) \text{ for } 0 \leq x \leq L \text{ if and only if} \]
\[ f(x) \text{ is continuous and both } f(0) = 0 \text{ and } f(L) = 0. \]

If \( f(0) \neq 0 \), then the odd extension of \( f(x) \) will have a jump discontinuity at \( x = 0 \), as illustrated in Figs. 3.3.19a and c. If \( f(L) \neq 0 \), then the odd extension at \( x = -L \) will be of opposite sign from \( f(L) \). Thus, the periodic extension will not be continuous at the endpoints if \( f(L) \neq 0 \) as in Figs. 3.3.19a and b.

**EXERCISES 3.3**

3.3.1. For the following functions, sketch \( f(x) \), the Fourier series of \( f(x) \), the Fourier sine series of \( f(x) \), and the Fourier cosine series of \( f(x) \).
3.3.2. For the following functions, sketch the Fourier sine series of \( f(x) \) and determine its Fourier coefficients.

(a) \( f(x) = 1 \)  

(b) \( f(x) = 1 + x \)

(c) \( f(x) = \begin{cases} x & x < 0 \\ 1 + x & x > 0 \end{cases} \)  

(d) \( f(x) = e^x \)

(e) \( f(x) = \begin{cases} 2 & x < 0 \\ e^{-x} & x > 0 \end{cases} \)

3.3.3. For the following functions, sketch the Fourier sine series of \( f(x) \). Also, roughly sketch the sum of a finite number of nonzero terms (at least the first two) of the Fourier sine series:

(a) \( f(x) = \cos \pi x / L \) [Verify formula (3.3.13).]  

(b) \( f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases} \)

(c) \( f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases} \)  

(d) \( f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases} \)

3.3.4. Sketch the Fourier cosine series of \( f(x) = \sin \pi x / L \). Briefly discuss.

3.3.5. For the following functions, sketch the Fourier cosine series of \( f(x) \) and determine its Fourier coefficients:

(a) \( f(x) = x^2 \)  

(b) \( f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases} \)

(c) \( f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases} \)

3.3.6. For the following functions, sketch the Fourier cosine series of \( f(x) \). Also, roughly sketch the sum of a finite number of nonzero terms (at least the first two) of the Fourier cosine series:

(a) \( f(x) = x \) [Use formulas (3.3.22) and (3.3.23).]  

(b) \( f(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases} \) [Use carefully formulas (3.2.6) and (3.2.7).]

(c) \( f(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases} \) [Hint: Add the functions in parts (b) and (c).]

3.3.7. Show that \( e^x \) is the sum of an even and an odd function.
3.3.8. (a) Determine formulas for the even extension of any \( f(x) \). Compare to the formula for the even part of \( f(x) \).
(b) Do the same for the odd extension of \( f(x) \) and the odd part of \( f(x) \).
(c) Calculate and sketch the four functions of parts (a) and (b) if

\[
f(x) = \begin{cases} 
  x & x > 0 \\
  x^2 & x < 0.
\end{cases}
\]

Graphically add the even and odd parts of \( f(x) \). What occurs? Similarly, add the even and odd extensions. What occurs then?

3.3.9. What is the sum of the Fourier sine series of \( f(x) \) and the Fourier cosine series of \( f(x) \)? [What is the sum of the even and odd extensions of \( f(x) \) ?]

3.3.10. If \( f(x) = \begin{cases} 
  x^2 & x < 0 \\
  e^{-x} & x > 0
\end{cases} \), what are the even and odd parts of \( f(x) \)?

3.3.11. Given a sketch of \( f(x) \), describe a procedure to sketch the even and odd parts of \( f(x) \).

3.3.12. (a) Graphically show that the even terms \((n \text{ even})\) of the Fourier sine series of any function on \( 0 \leq x \leq L \) are odd (antisymmetric) around \( x = L/2 \).
(b) Consider a function \( f(x) \) that is odd around \( x = L/2 \). Show that the odd coefficients \((n \text{ odd})\) of the Fourier sine series of \( f(x) \) on \( 0 \leq x \leq L \) are zero.

3.3.13. Consider a function \( f(x) \) that is even around \( x = L/2 \). Show that the even coefficients \((n \text{ even})\) of the Fourier sine series of \( f(x) \) on \( 0 \leq x \leq L \) are zero.

3.3.14. (a) Consider a function \( f(x) \) that is even around \( x = L/2 \). Show that the odd coefficients \((n \text{ odd})\) of the Fourier cosine series of \( f(x) \) on \( 0 \leq x \leq L \) are zero.
(b) Explain the result of part (a) by considering a Fourier cosine series of \( f(x) \) on the interval \( 0 \leq x \leq L/2 \).

3.3.15. Consider a function \( f(x) \) that is odd around \( x = L/2 \). Show that the even coefficients \((n \text{ even})\) of the Fourier cosine series of \( f(x) \) on \( 0 \leq x \leq L \) are zero.

3.3.16. Fourier series can be defined on other intervals besides \(-L \leq x \leq L\). Suppose that \( g(y) \) is defined for \( a \leq y \leq b \). Represent \( g(y) \) using periodic trigonometric functions with period \( b - a \). Determine formulas for the coefficients. [Hint: Use the linear transformation

\[
y = \frac{a+b}{2} + \frac{b-a}{2L} x.
\]
3.3.17. Consider
\[ \int_{0}^{1} \frac{dx}{1 + x^2}. \]
(a) Evaluate explicitly.
(b) Use the Taylor series of \(1/(1 + x^2)\) (itself a geometric series) to obtain
an infinite series for the integral.
(c) Equate part (a) to part (b) in order to derive a formula for \(x\).

3.3.18. For continuous functions,
(a) Under what conditions does \(f(x)\) equal its Fourier series for all \(x, -L \leq x \leq L\)?
(b) Under what conditions does \(f(x)\) equal its Fourier sine series for all \(x, 0 \leq x \leq L\)?
(c) Under what conditions does \(f(x)\) equal its Fourier cosine series for all \(x, 0 \leq x \leq L\)?

### 3.4 Term-by-Term Differentiation of Fourier Series

In solving partial differential equations by the method of separation of variables, the
homogeneous boundary conditions sometimes suggest that the desired solution is
either an infinite series of sines or cosines. For example, we consider one-dimensional
heat conduction with zero boundary conditions. As before, we want to solve the
initial boundary value problem

\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \]

(3.4.1)

\[ u(0, t) = 0, \quad u(L, t) = 0, \quad u(x, 0) = f(x). \]

(3.4.2)

By the method of separation of variables combined with the principle of superposition
(taking a finite linear combination of solutions), we know that

\[ u(x, t) = \sum_{n=1}^{N} B_n \sin \frac{n\pi x}{L} e^{-\left(n\pi/L\right)^2 kt} \]

solves the partial differential equation and the two homogeneous boundary conditions. To satisfy the initial conditions, in general an infinite series is needed. Does the infinite series

\[ u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\left(n\pi/L\right)^2 kt} \]

(3.4.3)
satisfy our problem? The theory of Fourier sine series shows that the Fourier co-
efficients \(B_n\) can be determined to satisfy any (piecewise smooth) initial condition
3.4. Term-by-Term Differentiation

[i.e., \( B_n = 2/L \int_0^L f(x) \sin n\pi x/L \, dx \)] To see if the infinite series actually satisfies the partial differential equation, we substitute (3.4.3) into (3.4.1). If the infinite Fourier series can be differentiated term by term, then

\[
\frac{\partial u}{\partial t} = -\sum_{n=1}^{\infty} k \left( \frac{n\pi}{L} \right)^2 B_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}
\]

and

\[
\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right)^2 B_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}.
\]

Thus, the heat equation \((\partial u/\partial t = k\partial^2 u/\partial x^2)\) is satisfied by the infinite Fourier series obtained by the method of separation of variables, if term-by-term differentiation of a Fourier series is valid.

**Term-by-term differentiation of infinite series.** Unfortunately, infinite series (even convergent infinite series) cannot always be differentiated term by term. It is not always true that

\[
\frac{d}{dx} \sum_{n=1}^{\infty} c_n u_n = \sum_{n=1}^{\infty} c_n \frac{du_n}{dx};
\]

the interchange of operations of differentiation and infinite summation is not always justified. However, we will find that in solving partial differential equations, all the procedures we have performed on the infinite Fourier series are valid. We will state and prove some needed theorems concerning the validity of term-by-term differentiation of just the type of Fourier series that arise in solving partial differential equations.

**Counterexample.** Even for Fourier series, term-by-term differentiation is not always valid. To illustrate the difficulty in term-by-term differentiation, consider the Fourier sine series of \(x\) (on the interval \(0 \leq x \leq L\)) sketched in Fig. 3.4.1:

\[
x = 2 \sum_{n=1}^{\infty} \frac{L}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{L}, \quad 0 \leq x < L,
\]

as obtained earlier [see (3.3.11) and (3.3.12)]. If we differentiate the function on the left-hand side, then we have the function 1. However, if we formally differentiate term by term the function on the right, then we arrive at

\[
2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos \frac{n\pi x}{L}.
\]

This is a cosine series, but it is not the cosine series of \(f(x) = 1\) (the cosine series of 1 is just 1). Thus, Fig. 3.4.1 is an example where we cannot differentiate term by term.\footnote{In addition, the resulting infinite series do not even converge anywhere, since the nth term does not approach zero.}
Chapter 3. Fourier Series

![Diagram](image)

**Figure 3.4.1** Fourier sine series of \( f(x) = x \).

**Fourier series.** We claim that this difficulty occurs any time the Fourier series of \( f(x) \) has a jump discontinuity. Term-by-term differentiation is not justified in these situations. Instead, we claim (and prove in an exercise) that

A Fourier series that is continuous can be differentiated term by term if \( f'(x) \) is piecewise smooth.

An alternative form of this theorem is written if we remember the condition for the Fourier series to be continuous:

If \( f(x) \) is piecewise smooth, then the Fourier series of a continuous function \( f(x) \) can be differentiated term by term if \( f(-L) = f(L) \).

The result of term-by-term differentiation is the Fourier series of \( f'(x) \), which may not be continuous. Similar results for sine and cosine series are of more frequent interest to the solution of our partial differential equations.

**Fourier cosine series.** For Fourier cosine series,

If \( f'(x) \) is piecewise smooth, then a continuous Fourier cosine series of \( f(x) \) can be differentiated term by term.

The result of term-by-term differentiation is the Fourier sine series of \( f'(x) \), which may not be continuous. Recall that \( f(x) \) only needs to be continuous for its Fourier cosine series to be continuous. Thus, this theorem can be stated in the following alternative form:

If \( f'(x) \) is piecewise smooth, then the Fourier cosine series of a continuous function \( f(x) \) can be differentiated term by term.
These statements apply to the Fourier cosine series of \( f(x) \):

\[
f(x) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}, \quad 0 \leq x \leq L,
\] 

(3.4.4)

where the = sign means that the infinite series converges to \( f(x) \) for all \( x \) \((0 \leq x \leq L)\) since \( f(x) \) is continuous. Mathematically, these theorems state that term-by-term differentiation is valid,

\[
f'(x) \sim - \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right) A_n \sin \frac{n\pi x}{L},
\] 

(3.4.5)

where \( \sim \) means equality where the Fourier sine series of \( f'(x) \) is continuous and means the series converges to the average where the Fourier sine series of \( f'(x) \) is discontinuous.

**Example.** Consider the Fourier cosine series of \( x \) [see (3.3.21), (3.3.22), and (3.3.23)],

\[
x = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n \text{ odd only}}^{\infty} \frac{1}{n^2} \cos \frac{n\pi x}{L}, \quad 0 \leq x \leq L,
\] 

(3.4.6)

as sketched in Fig. 3.4.2. Note the continuous nature of this series for \( 0 \leq x \leq L \), which results in the = sign in (3.4.6). The derivative of this Fourier cosine series is sketched in Fig. 3.4.3: it is the Fourier sine series of \( f(x) = 1 \). The Fourier sine series of \( f(x) = 1 \) can be obtained by term-by-term differentiation of the Fourier cosine series of \( f(x) = x \). Assuming that term-by-term differentiation of (3.4.6) is valid as claimed, it follows that

\[
1 \sim \frac{4}{\pi} \sum_{n \text{ odd only}}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{L},
\] 

(3.4.7)
which is in fact correct [see (3.3.8)].

**Fourier sine series.** A similar result is valid for Fourier sine series:

If \( f'(x) \) is piecewise smooth, then a continuous Fourier sine series of \( f(x) \) can be differentiated term by term.

However, if \( f(x) \) is continuous, then the Fourier sine series is continuous only if \( f(0) = 0 \) and \( f(L) = 0 \). Thus, we must be careful in differentiating term by term a Fourier sine series. In particular,

If \( f'(x) \) is piecewise smooth, then the Fourier sine series of a continuous function \( f(x) \) can only be differentiated term by term if \( f(0) = 0 \) and \( f(L) = 0 \).

**Proofs.** The proofs of these theorems are all quite similar. We include one since it provides a way to learn more about Fourier series and their differentiability. We will prove the validity of term-by-term differentiation of the Fourier sine series of a continuous function \( f(x) \), in the case when \( f'(x) \) is piecewise smooth and \( f(0) = 0 = f(L) \):

\[
f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L},
\]

(3.4.8)

where \( B_n \) are expressed below. An equality holds in (3.4.8) only if \( f(0) = 0 = f(L) \). If \( f'(x) \) is piecewise smooth, then \( f'(x) \) has a Fourier cosine series

\[
f'(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L},
\]

(3.4.9)

where \( A_0 \) and \( A_n \) are expressed in (3.4.10) and (3.4.11). This series will not converge to \( f'(x) \) at points of discontinuity of \( f'(x) \). We will have succeeded in showing a Fourier sine series may be term-by-term differentiated if we can verify that

\[
f'(x) \sim \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right) B_n \cos \frac{n\pi x}{L}
\]

\[
= A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right) B_n \cos \frac{n\pi x}{L}.
\]
[i.e., if $A_0 = 0$ and $A_n = (n\pi/L)B_n$.] The Fourier cosine series coefficients are derived from (3.4.9). If we integrate by parts, we obtain

$$
A_0 = \frac{1}{L} \int_0^L f'(x) \, dx = \frac{1}{L} \left[ f(L) - f(0) \right] \tag{3.4.10}
$$

$$(n \neq 0) \quad A_n = \frac{2}{L} \int_0^L f'(x) \cos \frac{n\pi x}{L} \, dx = \frac{2}{L} \left[ f(x) \cos \frac{n\pi x}{L} \right]_0^L + \frac{n\pi}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \tag{3.4.11}
$$

But from (3.4.8), $B_n$ is the Fourier sine series coefficient of $f(x)$,

$$
B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx,
$$

and thus for $n \neq 0$

$$
A_n = \frac{n\pi}{L} B_n + \frac{2}{L} \left[ (-1)^n f(L) - f(0) \right]. \tag{3.4.12}
$$

We thus see by comparing Fourier cosine coefficients that the Fourier sine series can be term-by-term differentiated only if both $f(L) - f(0) = 0$ (so that $A_0 = 0$) and $(-1)^n f(L) - f(0) = 0$ [so that $A_n = (n\pi/L)B_n$.] Both of these conditions hold only if

$$
f(0) = f(L) = 0,
$$

exactly the conditions for a Fourier sine series of a continuous function to be continuous. Thus, we have completed the proof. However, this demonstration has given us more information. Namely, it gives the formula to differentiate the Fourier sine series of a continuous function when the series is not continuous. We have that

If $f'(x)$ is piecewise smooth, then the Fourier sine series of a continuous function $f(x)$,

$$
f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}
$$

cannot, in general be differentiated term by term. However,

$$
f'(x) \sim \frac{1}{L} \left[ f(L) - f(0) \right] + \sum_{n=1}^{\infty} \left[ \frac{n\pi}{L} B_n + \frac{2}{L} \left[ (-1)^n f(L) - f(0) \right] \right] \cos \frac{n\pi x}{L}.
$$

\hspace{1cm} (3.4.13)

In this proof, it may appear that we never needed $f(x)$ to be continuous. However, we applied integration by parts in order to derive (3.4.9). In the usual presentation in calculus, integration by parts is stated as being valid if both $u(x)$ and
v(x) and their derivatives are continuous. This is overly restrictive for our work. As is clarified somewhat in an exercise, we state that integration by parts is valid if only u(x) and v(x) are continuous. It is not necessary for their derivatives to be continuous. Thus the result of integration by parts is valid only if f(x) is continuous.

**Example.** Let us reconsider the Fourier sine series of f(x) = x,

\[ x \sim 2 \sum_{n=1}^{\infty} \frac{L}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{L}. \]  

We already know that (d/dx)x = 1 does not have a Fourier cosine series that results from term-by-term differentiation of (3.4.14) since f(L) ≠ 0. However, (3.4.13) may be applied since f(x) is continuous [and f′(x) is piecewise smooth]. Noting that f(0) = 0, f(L) = L and (n\pi/L)B_n = 2(-1)^{n+1}, it follows that the Fourier cosine series of df/dx is

\[ \frac{df}{dx} \sim 1. \]

The constant function 1 is exactly the Fourier cosine series of df/dx since f = x implies that df/dx = 1. Thus, the r.h.s. of (3.4.13) gives the correct expression for the Fourier cosine series of f′(x) when the Fourier sine series of f(x) is known, even if f(0) ≠ 0 and/or f(L) ≠ 0.

**Method of eigenfunction expansion.** Let us see how our results concerning the conditions under which a Fourier series may be differentiated term by term may be applied to our study of partial differential equations. We consider the heat equation (3.4.1) with zero boundary conditions at x = 0 and x = L. We will show that (3.4.3) is the correct infinite series representation of the solution of this problem. We will show this by utilizing an alternative scheme to obtain (3.4.3) known as the method of eigenfunction expansion, whose importance is that it may also be used when there are sources or the boundary conditions are not homogeneous (see Exercises 3.4.9–3.4.12 and Chapter 7). We begin by assuming that we have a solution u(x, t) that is continuous such that ∂u/∂t, ∂u/∂x and ∂²u/∂x² are also continuous. Now we expand the unknown solution u(x, t) in terms of the eigenfunctions of the problem (with homogeneous boundary conditions). In this example, the eigenfunctions are \sin n\pi x/L, suggesting a Fourier sine series for each time:

\[ u(x, t) \sim \sum_{n=1}^{\infty} B_n(t) \sin \frac{n\pi x}{L}; \]  

the Fourier sine coefficients B_n will depend on time, B_n(t).

The initial condition [u(x, 0) = f(x)] is satisfied if

\[ f(x) \sim \sum_{n=1}^{\infty} B_n(0) \sin \frac{n\pi x}{L}, \]  

(3.4.15)
determining the Fourier sine coefficient initially
\[ B_n(0) = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx. \]  
(3.4.17)

All that remains is to investigate whether the Fourier sine series representation of \( u(x,t) \), (3.4.15), can satisfy the heat equation, \( \partial u/\partial t = k \partial^2 u/\partial x^2 \). To do that we must differentiate the Fourier sine series. It is here that our results concerning term-by-term differentiation are useful.

First we need to compute two derivatives with respect to \( x \). If \( u(x,t) \) is continuous, then the Fourier sine series of \( u(x,t) \) can be differentiated term by term if \( u(0,t) = 0 \) and \( u(L,t) = 0 \). Since these are exactly the boundary conditions on \( u(x,t) \), it follows from (3.4.15) that
\[ \frac{\partial u}{\partial x} \sim \sum_{n=1}^{\infty} \frac{n\pi}{L} B_n(t) \cos \frac{n\pi x}{L}. \]  
(3.4.18)

Since \( \partial u/\partial x \) is also assumed to be continuous, an equality holds in (3.4.18). Furthermore, the Fourier cosine series of \( \partial u/\partial x \) can now be term-by-term differentiated, yielding
\[ \frac{\partial^2 u}{\partial x^2} \sim -\sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right)^2 B_n(t) \sin \frac{n\pi x}{L}. \]  
(3.4.19)

Note the importance of the separation of variables solution. Sines were differentiated at the stage in which the boundary conditions occurred that allowed sines to be differentiated. Cosines occurred with no boundary condition, consistent with the fact that a Fourier cosine series does not need any subsidiary conditions in order to be differentiated. To complete the substitution of the Fourier sine series into the partial differential equation, we need only to compute \( \partial u/\partial t \). If we can also term-by-term differentiate with respect to \( t \), then
\[ \frac{\partial u}{\partial t} \sim \sum_{n=1}^{\infty} \frac{dB_n}{dt} \sin \frac{n\pi x}{L}. \]  
(3.4.20)

If this last term-by-term differentiation is justified, we see that the Fourier sine series (3.4.15) solves the partial differential equation if
\[ \frac{dB_n}{dt} = -k \left( \frac{n\pi}{L} \right)^2 B_n(t). \]  
(3.4.21)

The Fourier sine coefficient \( B_n(t) \) satisfies a first-order linear differential equation with constant coefficients. The solution of (3.4.21) is
\[ B_n(t) = B_n(0) e^{-\left(\frac{n\pi}{L}\right)^2 kt}, \]
where \( B_n(0) \) is given by (3.4.17). Thus, we have derived that (3.4.3) is valid, justifying the method of separation of variables.

Can we justify term-by-term differentiation with respect to the parameter \( t \)? The following theorem states the conditions under which this operation is valid:
The Fourier series of a continuous function \( u(x, t) \) (depending on a parameter \( t \))
\[
u(x, t) = a_0(t) + \sum_{n=1}^{\infty} \left[ a_n(t) \cos \frac{n\pi x}{L} + b_n(t) \sin \frac{n\pi x}{L} \right]
\]
can be differentiated term by term with respect to the parameter \( t \), yielding
\[
\frac{\partial}{\partial t} u(x, t) \sim a'_0(t) + \sum_{n=1}^{\infty} \left[ a'_n(t) \cos \frac{n\pi x}{L} + b'_n(t) \sin \frac{n\pi x}{L} \right]
\]
if \( \partial u/\partial t \) is piecewise smooth.

We omit its proof (see Exercise 3.4.7), which depends on the fact that
\[
\frac{\partial}{\partial t} \int_{-L}^{L} g(x, t) \, dx = \int_{-L}^{L} \frac{\partial g}{\partial t} \, dx
\]
is valid if \( g \) is continuous.

In summary, we have verified that the Fourier sine series is actually a solution of the heat equation satisfying the boundary conditions \( u(0, t) = 0 \) and \( u(L, t) = 0 \). Now we have two reasons for choosing a Fourier sine series for this problem. First, the method of separation of variables implies that if \( u(0, t) = 0 \) and \( u(L, t) = 0 \), then the appropriate eigenfunctions are \( \sin \frac{n\pi x}{L} \). Second, we now see that all the differentiations of the infinite sine series are justified, where we need to assume that \( u(0, t) = 0 \) and \( u(L, t) = 0 \), exactly the physical boundary conditions.

EXERCISES 3.4

3.4.1. The integration-by-parts formula
\[
\int_{a}^{b} u \frac{dv}{dx} \, dx = uv \bigg|_{a}^{b} - \int_{a}^{b} u \frac{dv}{dx} \, dx
\]
is known to be valid for functions \( u(x) \) and \( v(x) \), which are continuous and have continuous first derivatives. However, we will assume that \( u, v, du/dx, \) and \( dv/dx \) are continuous only for \( a \leq x \leq c \) and \( c \leq x \leq b \); we assume that all quantities may have a jump discontinuity at \( x = c \).

*(a) Derive an expression for \( \int_{a}^{b} u \, dv/dx \, dx \) in terms of \( \int_{a}^{b} v \, du/dx \, dx \).

(b) Show that this reduces to the integration-by-parts formula if \( u \) and \( v \) are continuous across \( x = c \). It is not necessary for \( du/dx \) and \( dv/dx \) to be continuous at \( x = c \).

3.4.2. Suppose that \( f(x) \) and \( df/dx \) are piecewise smooth. Prove that the Fourier series of \( f(x) \) can be differentiated term by term if the Fourier series of \( f(x) \) is continuous.
3.4. Term-by-Term Differentiation

3.4.3. Suppose that $f(x)$ is continuous [except for a jump discontinuity at $x = x_0$, $f(x_0^-) = \alpha$ and $f(x_0^+) = \beta$] and $df/dx$ is piecewise smooth.

*(a) Determine the Fourier sine series of $df/dx$ in terms of the Fourier cosine series coefficients of $f(x)$.

(b) Determine the Fourier cosine series of $df/dx$ in terms of the Fourier sine series coefficients of $f(x)$.

3.4.4. Suppose that $f(x)$ and $df/dx$ are piecewise smooth.

(a) Prove that the Fourier sine series of a continuous function $f(x)$ can only be differentiated term by term if $f(0) = 0$ and $f(L) = 0$.

(b) Prove that the Fourier cosine series of a continuous function $f(x)$ can be differentiated term by term.

3.4.5. Using (3.3.13) determine the Fourier cosine series of $\sin \pi x/L$.

3.4.6. There are some things wrong in the following demonstration. Find the mistakes and correct them.

In this exercise we attempt to obtain the Fourier cosine coefficients of $e^x$:

$$e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}.$$  \hspace{1cm} (3.4.22)

Differentiating yields

$$e^x = -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin \frac{n\pi x}{L},$$

the Fourier sine series of $e^x$. Differentiating again yields

$$e^x = -\sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right)^2 A_n \cos \frac{n\pi x}{L}.$$  \hspace{1cm} (3.4.23)

Since equations (3.4.22) and (3.4.23) give the Fourier cosine series of $e^x$, they must be identical. Thus,

$$\begin{align*}
A_0 &= 0 \\
A_n &= 0
\end{align*}$$

(Obviously wrong!)

By correcting the mistakes, you should be able to obtain $A_0$ and $A_n$ without using the typical technique, that is, $A_n = 2/L \int_0^L e^x \cos n\pi x/L \, dx$.

3.4.7. Prove that the Fourier series of a continuous function $u(x,t)$ can be differentiated term by term with respect to the parameter $t$ if $\partial u/\partial t$ is piecewise smooth.
3.4.8. Consider
\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \]
subject to
\[ \partial u / \partial x(0, t) = 0, \quad \partial u / \partial x(L, t) = 0, \quad \text{and} \quad u(x, 0) = f(x). \]

Solve in the following way. Look for the solution as a Fourier cosine series. Assume that \( u \) and \( \partial u / \partial x \) are continuous and \( \partial^2 u / \partial x^2 \) and \( \partial u / \partial t \) are piecewise smooth. Justify all differentiations of infinite series.

*3.4.9 Consider the heat equation with a known source \( q(x, t) \):
\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + q(x, t) \quad \text{with} \quad u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0. \]

Assume that \( q(x, t) \) (for each \( t > 0 \)) is a piecewise smooth function of \( x \). Also assume that \( u \) and \( \partial u / \partial x \) are continuous functions of \( x \) (for \( t > 0 \)) and \( \partial^2 u / \partial x^2 \) and \( \partial u / \partial t \) are piecewise smooth. Thus,
\[ u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L}. \]

What ordinary differential equation does \( b_n(t) \) satisfy? Do not solve this differential equation.

3.4.10. Modify Exercise 3.4.9 if instead \( \partial u / \partial x(0, t) = 0 \) and \( \partial u / \partial x(L, t) = 0 \).

3.4.11. Consider the nonhomogeneous heat equation (with a steady heat source):
\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + g(x). \]

Solve this equation with the initial condition
\[ u(x, 0) = f(x) \]

and the boundary conditions
\[ u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0. \]

Assume that a continuous solution exists (with continuous derivatives). [Hints: Expand the solution as a Fourier sine series (i.e., use the method of eigenfunction expansion). Expand \( g(x) \) as a Fourier sine series. Solve for the Fourier sine series of the solution. Justify all differentiations with respect to \( x \).]

*3.4.12. Solve the following nonhomogeneous problem:
\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + e^{-t} + e^{-2t} \cos \frac{3\pi x}{L} \quad \text{[assume that} 2 \neq k(3\pi/L)^2].} \]
subject to
\[ \frac{\partial u}{\partial x}(0, t) = 0, \quad \frac{\partial u}{\partial x}(L, t) = 0, \quad \text{and} \quad u(x, 0) = f(x). \]

Use the following method. Look for the solution as a Fourier cosine series. Justify all differentiations of infinite series (assume appropriate continuity).

3.4.13. Consider
\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \]
subject to
\[ u(0, t) = A(t), \quad u(L, t) = 0, \quad \text{and} \quad u(x, 0) = g(x). \]

Assume that \( u(x, t) \) has a Fourier sine series. Determine a differential equation for the Fourier coefficients (assume appropriate continuity).

### 3.5 Term-By-Term Integration of Fourier Series

In doing mathematical manipulations with infinite series, we must remember that some properties of finite series do not hold for infinite series. In particular, Sec. 3.4 indicated that we must be especially careful differentiating term by term an infinite Fourier series. The following theorem however, enables us to integrate Fourier series without caution:

A Fourier series of piecewise smooth \( f(x) \) can always be integrated term by term and the result is a convergent infinite series that always converges to the integral of \( f(x) \) for \( -L \leq x \leq L \) (even if the original Fourier series has jump discontinuities).

Remarkably, the new series formed by term-by-term integration is continuous. However, the new series may not be a Fourier series.

To quantify this statement, let us suppose that \( f(x) \) is piecewise smooth and hence has a Fourier series in the range \(-L \leq x \leq L\) (not necessarily continuous):

\[ f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \quad (3.5.1) \]

We will prove our claim that we can just integrate this result term by term:

\[ \int_{-L}^{x} f(t) \, dt \sim a_0 (x + L) + \sum_{n=1}^{\infty} \left( a_n \int_{-L}^{x} \cos \frac{n\pi t}{L} \, dt + b_n \int_{-L}^{x} \sin \frac{n\pi t}{L} \, dt \right). \]
Performing the indicated integration yields

$$\int_{-L}^{x} f(t) \, dt \sim a_0(x + L) + \sum_{n=1}^{\infty} \left[ \frac{a_n}{n\pi/L} \sin \frac{n\pi x}{L} + \frac{b_n}{n\pi/L} \left( \cos n\pi - \cos \frac{n\pi x}{L} \right) \right].$$

(3.5.2)

We will actually show that the preceding statement is valid with an = sign. If term-by-term integration from $-L$ to $x$ of a Fourier series is valid, then any definite integration is also valid since

$$\int_{a}^{b} = \int_{-L}^{b} - \int_{-L}^{a}.$$

Example. Term-by-term integration has some interesting applications. Recall that the Fourier sine series for $f(x) = 1$ is given by

$$1 \sim \frac{4}{\pi} \left( \sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi x}{L} + \cdots \right),$$

(3.5.3)

where $\sim$ is used since (3.5.3) is an equality only for $0 < x < L$. Integrating term by term from 0 to $x$ results in

$$x \sim \frac{4L}{\pi^2} \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right) - \frac{4L}{\pi^2} \left( \cos \frac{\pi x}{L} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots \right),$$

$$0 \leq x \leq L,$$

(3.5.4)

where because of our theorem the = sign can be used. We immediately recognize that (3.5.4) should be the Fourier cosine series of the function $x$. It was obtained by integrating the Fourier sine series of $f(x) = 1$. However, an infinite series of constants appears in (3.5.4); it is the constant term of the Fourier cosine series of $x$. In this way we can evaluate that infinite series,

$$\frac{4L}{\pi^2} \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right) = \frac{1}{L} \int_{0}^{L} x \, dx = \frac{1}{2} L.$$

Thus, we obtain the usual form for the Fourier cosine series for $x$,

$$x = \frac{L}{2} - \frac{4L}{\pi^2} \left( \cos \frac{\pi x}{L} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots \right), 0 \leq x \leq L.$$

(3.5.5)

The process of deriving new series from old ones can be continued. Integrating (3.5.5) from 0 to $x$ yields

$$\frac{x^2}{2} = \frac{L}{2} x - \frac{4L^2}{\pi^3} \left( \sin \frac{\pi x}{L} + \frac{\sin 3\pi x}{3^3} + \frac{\sin 5\pi x}{5^3} + \cdots \right).$$

(3.5.6)

This example illustrates that integrating a Fourier series term by term does not necessarily yield another Fourier series. However, (3.5.6) can be looked at as either yielding
1. The Fourier sine series of \( x^2/2 - (L/2)x \), or

2. The Fourier sine series of \( x^2/2 \), where the Fourier sine series of \( x \) is needed first [see (3.3.11) and (3.3.12)].

An alternative procedure is to perform indefinite integration. In this case an arbitrary constant must be included and evaluated. For example, reconsider the Fourier sine series of \( f(x) = 1 \), (3.5.3). By term-by-term indefinite integration we derive the Fourier cosine series of \( x \),

\[
x = c - \frac{4L}{\pi^2} \left( \cos \frac{\pi x}{L} + \frac{\cos 3\pi x/L}{3^2} + \frac{\cos 5\pi x/L}{5^2} + \cdots \right).
\]

The constant of integration is not arbitrary; it must be evaluated. Here \( c \) is again the constant term of the Fourier cosine series of \( x \), \( c = (1/L) \int_0^L x \, dx = L/2 \).

**Proof on integrating Fourier series.** Consider

\[
F(x) = \int_{-L}^{x} f(t) \, dt.
\]  

(3.5.7)

This integral is a continuous function of \( x \) since \( f(x) \) is piecewise smooth. \( F(x) \) has a continuous Fourier series only if \( F(L) = F(-L) \) [otherwise, remember that the periodic nature of the Fourier series implies that the Fourier series does not converge to \( F(x) \) at the endpoints \( x = \pm L \)]. However, note that from the definition (3.5.7),

\[
F(-L) = 0 \quad \text{and} \quad F(L) = \int_{-L}^{L} f(t) \, dt = 2La_0.
\]

Thus, in general \( F(x) \) does not have a continuous Fourier series. In Fig. 3.5.1, \( F(x) \) is sketched, illustrating the fact that usually \( F(-L) \neq F(L) \). However, consider the straight line connecting the point \( F(-L) \) to \( F(L) \), \( y = a_0(x + L) \). \( G(x) \), defined to be the difference between \( F(x) \) and the straight line,

\[
G(x) \equiv F(x) - a_0(x + L),
\]  

(3.5.8)

will be zero at both ends, \( x = \pm L \),

\[
G(-L) = G(L) = 0,
\]

as illustrated in Fig. 3.5.1. \( G(x) \) is also continuous. Thus, \( G(x) \) satisfies the properties that enable the Fourier series of \( G(x) \) actually to equal \( G(x) \):

\[
G(x) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right),
\]  

(3.5.9)

where the = sign is emphasized. These Fourier coefficients can be computed as

\[
A_n = \frac{1}{L} \int_{-L}^{L} [F(x) - a_0(x + L)] \cos \frac{n\pi x}{L} \, dx \quad (n \neq 0).
\]
The $x$-term can be dropped since it is odd (i.e., $\int_{-L}^{L} x \cos n\pi x/L \, dx = 0$). The resulting expression can be integrated by parts as follows:

$$u = F(x) - a_0 L \quad \quad \quad dv = \cos \frac{n\pi x}{L} \, dx$$

$$du = \frac{dF}{dx} \, dx = f(x) \, dx \quad \quad \quad v = \frac{L}{n\pi} \sin \frac{n\pi x}{L},$$

yielding

$$A_n = \frac{1}{L} \left[ (F(x) - a_0 L) \frac{\sin n\pi x}{n\pi} \left. \right|_{-L}^{L} - \frac{L}{n\pi} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx \right] = -\frac{b_n}{n\pi / L}, \quad (3.5.10)$$

where we have recognized that $b_n$ is the Fourier sine coefficient of $f(x)$. In a similar manner (which we leave as an exercise), it can be shown that

$$B_n = \frac{a_n}{n\pi / L},$$

where $a_n$ is the Fourier cosine coefficient of $f(x)$. $A_0$ can be calculated in a different manner (the previous method will not work). Since $G(L) = 0$ and the Fourier series of $G(x)$ is pointwise convergent, from (3.5.9) it follows that

$$0 = A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi = A_0 - \sum_{n=1}^{\infty} \frac{b_n}{n\pi / L} \cos n\pi$$

since $A_n = -b_n/(n\pi / L)$. Thus, we have shown from (3.5.9) that

$$F(x) = a_0 (x + L) + \sum_{n=1}^{\infty} \left[ \frac{a_n}{n\pi / L} \sin \frac{n\pi x}{L} + \frac{b_n}{n\pi / L} \left( \cos n\pi \sin \frac{n\pi x}{L} \right) \right], \quad (3.5.11)$$

exactly the result of simple term-by-term integration. However, notice that (3.5.11) is not the Fourier series of $F(x)$, since $a_0 x$ appears. Nonetheless, (3.5.11) is valid. We have now justified term-by-term integration of Fourier series.
EXERCISES 3.5

3.5.1. Consider

\[ x^2 \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} . \]  \hspace{1cm} (3.5.12)

(a) Determine \( b_n \) from (3.3.11), (3.3.12), and (3.5.6).

(b) For what values of \( x \) is (3.5.12) an equality?

*(c) Derive the Fourier cosine series for \( x^3 \) from (3.5.12). For what values of \( x \) will this be an equality?

3.5.2. (a) Using (3.3.11) and (3.3.12), obtain the Fourier cosine series of \( x^2 \).

(b) From part (a), determine the Fourier sine series of \( x^3 \).

3.5.3. Generalize Exercise 3.5.2, in order to derive the Fourier sine series of \( x^m \), \( m \) odd.

*3.5.4. Suppose that \( \cosh x \sim \sum_{n=1}^{\infty} b_n \sin n\pi x/L \).

(a) Determine \( b_n \) by correctly differentiating this series twice.

(b) Determine \( b_n \) by integrating this series twice.

3.5.5. Show that \( B_n \) in (3.5.9) satisfies \( B_n = a_n/(n\pi/L) \), where \( a_n \) is defined by (3.5.1).

3.5.6. Evaluate

\[ 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots \]

by evaluating (3.5.5) at \( x = 0 \).

*3.5.7. Evaluate

\[ 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots \]

using (3.5.6).

3.6 Complex Form of Fourier Series

With periodic boundary conditions, we have found the theory of Fourier series to be quite useful:

\[ f(x) \sim a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) , \] \hspace{1cm} (3.6.1)
where

\[ a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx \]  
(3.6.2)

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx \]  
(3.6.3)

\[ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx. \]  
(3.6.4)

To introduce complex exponentials instead of sines and cosines, we use Euler's formulas

\[ \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \]

It follows that

\[ f(x) \sim a_0 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n - ib_n) e^{in\pi x/L} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n + ib_n) e^{-in\pi x/L}. \]  
(3.6.5)

In order to only have \( e^{-in\pi x/L} \), we change the dummy index in the first summation, replacing \( n \) by \(-n\). Thus,

\[ f(x) \sim a_0 + \frac{1}{2} \sum_{n=-1}^{\infty} [a_{-n} - ib_{-n}] e^{-in\pi x/L} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n + ib_n) e^{-in\pi x/L}. \]

From the definition of \( a_n \) and \( b_n \), (3.6.3) and (3.6.4), \( a_{-n} = a_n \) and \( b_{-n} = -b_n \). Thus, if we define

\[ c_0 = a_0 \]
\[ c_n = \frac{a_n + ib_n}{2}, \]

then \( f(x) \) becomes simply

\[ f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}. \]  
(3.6.6)

Equation (3.6.6) is known as the complex form of the Fourier series of \( f(x) \).\(^3\)

It is equivalent to the usual form. It is more compact to write, but it is only used infrequently. In this form the complex Fourier coefficients are

\[ c_n = \frac{1}{2} (a_n + ib_n) = \frac{1}{2L} \int_{-L}^{L} f(x) \left( \cos \frac{n\pi x}{L} + i \sin \frac{n\pi x}{L} \right) \, dx. \quad (n \neq 0) \]

\(^3\)As before, an equal sign appears if \( f(x) \) is continuous (and periodic, \( f(-L) = f(L) \)). At a jump discontinuity of \( f(x) \) in the interior, the series converges to \([f(x^+) + f(x^-)]/2\).
3.6. Complex Form

We immediately recognize a simplification, using Euler’s formula. Thus, we derive a formula for the complex Fourier coefficients

\[
c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{in\pi x/L} \, dx. \tag{3.6.7}
\]

(a)\( n \)

Notice that the complex Fourier series representation of \( f(x) \) has \( e^{-in\pi x/L} \) and is summed over the discrete integers corresponding to the sum over the discrete eigenvalues. The complex Fourier coefficients, on the other hand, involve \( e^{+in\pi x/L} \) and are integrated over the region of definition of \( f(x) \) (with periodic boundary conditions), namely \(-L \leq x \leq L\). If \( f(x) \) is real, \( c_{-n} = \overline{c_n} \) (see Exercise 3.6.2).

**Complex orthogonality**. There is an alternative way to derive the formula for the complex Fourier coefficients. Always, in the past, we have determined Fourier coefficients using the orthogonality of the eigenfunctions. A similar idea holds here. However, here the eigenfunctions \( e^{-in\pi x/L} \) are complex. For complex functions the concept of orthogonality must be slightly modified. A complex function \( \phi \) is said to be orthogonal to a complex function \( \psi \) (over an interval \( a \leq x \leq b \)) if \( \int_{a}^{b} \overline{\phi} \psi \, dx = 0 \), where \( \overline{\phi} \) is the complex conjugate of \( \phi \). This guarantees that the length squared of a complex function \( f \), defined by \( \int_{a}^{b} \overline{f} f \, dx \), is positive (this would not have been valid for \( \int_{a}^{b} f f \, dx \) since \( f \) is complex).

Using this notion of orthogonality, the eigenfunctions \( e^{-in\pi x/L} \), \(-\infty < n < \infty\), can be verified to form an orthogonal set because by simple integration

\[
\int_{-L}^{L} (e^{-im\pi x/L}) e^{-in\pi x/L} \, dx = \begin{cases} 
0 & n \neq m \\
2L & n = m,
\end{cases}
\]

since

\[
\overline{(e^{-im\pi x/L})} = e^{im\pi x/L}.
\]

Now to determine the complex Fourier coefficients \( c_n \), we multiply (3.6.6) by \( e^{in\pi x/L} \) and integrate from \(-L\) to \(+L\) (assuming that the term-by-term use of these operations is valid). In this way

\[
\int_{-L}^{L} f(x) e^{in\pi x/L} \, dx = \sum_{n=-\infty}^{\infty} c_n \int_{-L}^{L} e^{in\pi x/L} e^{-in\pi x/L} \, dx.
\]

Using the orthogonality condition, the sum reduces to one term, \( n = m \). Thus,

\[
\int_{-L}^{L} f(x) e^{in\pi x/L} \, dx = 2L c_m,
\]

which explains the \( 1/2L \) in (3.6.7) as well as the switch of signs in the exponent.
EXERCISES 3.6

*3.6.1. Consider

\[ f(x) = \begin{cases} 
0 & x < x_0 \\
1/\Delta & x_0 < x < x_0 + \Delta \\
0 & x > x_0 + \Delta. 
\end{cases} \]

Assume that \( x_0 > -L \) and \( x_0 + \Delta < L \). Determine the complex Fourier coefficients \( c_n \).

3.6.2. If \( f(x) \) is real, show that \( c_{-n} = \bar{c}_n \).