The status of the Scholze-Stix Report and an analysis of the Mochizuki-Scholze-Stix Controversy

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June 28, 2024

§ 1.1 The need for this document After Mochizuki's [Mochizuki, 2021a,b,c,d] preprints were released in 2012, there was tremendous excitement in the math community, and a serious effort was made to understand his work. Unfortunately, neither Mochizuki, nor any of his 'experts' could provide satisfactory answers or examples to explain his theory. So the Scholze-Stix Report ([Scholze and Stix, 2018]) which talked about the flaws in Mochizuki's work ([Mochizuki, 2021a,b,c,d]) was widely accepted as a valid conclusion.

However, I have always believed that if Mochizuki's proof has to be rejected, then it must be based on a complete understanding of its ideas and present correct mathematical arguments, precisely identifying the issue(s) with its claimed strategy.

Despite having discussed the flaws in [Scholze and Stix, 2018], [Scholze, 2021] in my papers and the clarity and precision that my work has brought to Mochizuki's work, [Scholze and Stix, 2018], [Scholze, 2021] are still considered by many as definitive conclusions on [Mochizuki, 2021a,b,c,d]. Hence the need for this document.

This document provides quantitative evidence (§ 1.2, § 1.3) that due to the lack of clarity in Mochizuki's work, [Scholze and Stix, 2018] understanding of Mochizuki's work is flawed at a fundamental level, leading to an inaccurate analysis of [Mochizuki, 2021a,b,c,d]. My works on this subject provide far more accurate critiques of [Mochizuki, 2021a,b,c,d] and how the highlighted issues can be circumvented.

I have tried my best to be fair in my critiques of [Mochizuki, 2021a,b,c,d] and [Scholze and Stix, 2018]. However, both have been reticent in publicly acknowledging my work. Taking on the claims of two powerful mathematicians, I completely understand that the professional fallout for me is substantial and academically debilitating (this has already played out as may be evident from Mochizuki's colorful language and analogies in rejecting my work while many arithmetic geometers have simply distanced themselves from the conversation). I believe that more voices should participate in the discussion of the abc-conjecture because the public comments by both Mochizuki and Scholze (and some others who have echoed Scholze) make it clear that they simply do not wish to be second-guessed on this matter. This has emboldened some mathematicians to publicly dismiss my work even though it brings many new ideas to the table.

The conclusion of this document is independent of the proof of the abc-conjecture.

§ 1.2 Debunking some popular myths surrounding Mochizuki's Theory I want to debunk some of the popular myths about Mochizuki's Theory.

Let me set out some notation for this discussion. Let E be a p-adic field and let X/E be a geometrically connected, smooth, hyperbolic curve over E and assume that X is definable over some number field. Mochizuki's theory works with tempered fundamental groups [André, 2003], [Lepage, 2010]. This requires some additional notation. Let $K \supset E$ be an algebraically closed, complete (rank one) valued field containing E isometrically [such a field is an algebraically closed, perfectoid field [Scholze, 2012]]. Let X_E^{an} be the Berkovich analytic space of X/E and let $* : \mathscr{M}(K) \to X_E^{an}$ be a K-valued point of the analytic space X_E^{an}. Such a point is a geometric base-point and allows one to compute the tempered fundamental group

$$\Pi^{temp}_{X/E;K} := \pi_1^{temp}(X_E^{an}, *_K : \mathscr{M}(K) \to X_E^{an})$$

using the K-geometric base-point $*_K : \mathscr{M}(K) \to X_E^{an}$. The isomorphism class of this group is independent of the chosen geometric base-point, so often one writes $\Pi^{temp}_{X/E}$ instead of $\Pi^{temp}_{X/E;K}$.

§ 1.2.1 Myth Mochizuki's [Mochizuki, 2021a,b,c,d] is about étale fundamental group of X/E.

Sources For This Myth May date back to Mochizuki's preprints in 2012, but also see [Fesenko, 2015], [Scholze and Stix, 2018], the discussion on Peter Woit's Blog [Scholze et al., April 2020].

This Myth is False Mochizuki's IUTT is about tempered fundamental groups and *Mochizuki's Key Principle of Inter-Universality* [Mochizuki, 2021a, § I3, Pages 25–26] requires us to remember the geometric base-points for tempered fundamental groups i.e. one works with $\Pi^{temp}_{X/E;K}$ and not $\Pi^{temp}_{X/E}$. [Note that Mochizuki does not use perfectoid fields and that is why I have objected to his papers. Once this information is thrown in, I show that there is a valid theory satisfying all the claimed properties.] [The table given below is taken from [Joshi, 2024b].]

Comparison of geometric base-points for étale and tempered fundamental groups

	étale fund. group	tempered fund. group
Input datum	$(X/\mathbb{Q}_p, *_K : \mathrm{Spec}(K) \to X)$ where $X/\mathbb{Q}_p$ is a scheme, $K \supset \mathbb{Q}_p$ is an alg. closed field, and $*_K : \mathrm{Spec}(K) \to X$ is a K-geom. base point of X	$(X^{an}_{\mathbb{Q}_p}, *_K : \mathscr{M}(K) \to X^{an}_{\mathbb{Q}_p})$ where $X^{an}_{\mathbb{Q}_p}$ is an analytic space over $\mathbb{Q}_p$, $K \supset \mathbb{Q}_p$ an alg. closed, perfectoid field, and $*_K : \mathscr{M}(K) \to X^{an}_{\mathbb{Q}_p}$ is a K-geom. base point of $X^{an}_{\mathbb{Q}_p}$.
	isom. class of alg. closed fields $K \supset \mathbb{Q}_p$ is determined by the cardinality of K	isom. class of alg. closed, perfectoid fields $K \supset \mathbb{Q}_p$ is not determined by cardinality of K.

Caution For important reasons, one works with an alg. closed perfectoid field K, together with tilting data i.e. one works with untilts rather than K. *So readers should not walk away with the impression that working with perfectoid fields is adequate. See [Joshi, 2022, Definition 5.1] for the precise datum required.*

Let me also remark that Mochizuki's Anabelian Reconstruction Theory methods of [Mochizuki, 2012, 2013, 2015], cannot provide reconstruction of the perfectoid field in the geometric base-point datum. For example, the algebraically closed perfectoid field $\mathbb{C}_p$, cannot be reconstructed from the topological absolute Galois group $G_{\mathbb{Q}_p}$ as was shown in [Mochizuki, 1997]. Hence dealing with the geometric base-point datum (as required by Mochizuki's Key Principle of Inter-Universality) would necessarily require taking the approach I have followed in my theory (and by doing so I obtain a canonical theory in which all of Mochizuki's ideas have intrinsic meaning). The failure of [Mochizuki, 2021a,b,c,d] to treat its own principle clearly and accurately forms the core of my mathematical objections to [Mochizuki, 2021a,b,c,d].

§ 1.2.2 Myth Validity of the Absolute Grothendieck Conjecture for hyperbolic curves implies Mochizuki's proof [Mochizuki, 2021a,b,c,d] is impossible.

Sources For This Myth Advertisement by Mochizuki (and others) about anabelian aspects of [Mochizuki, 2021a,b,c,d] (and use of [Mochizuki, 2015]), are a possible source of this myth (and in my opinion, Mochizuki failed to bring clarity on this point). But it may have taken strong roots because of [Scholze and Stix, 2018] Footnote 8.

This Myth is False Scholze-Stix conflated Teichmuller Theory and Moduli theory because of this belief. Existence of Teichmuller Theory of the sort Mochizuki claims (and established in my work) is independent of the validity of the Absolute Grothendieck Conjecture over number fields and p-adic fields. From my point of view the non-validity of the Absolute Grothendieck Conjecture over algebraically closed, perfectoid fields (or over $\mathbb{C}$) is a better way of establishing Mochizuki's Teichmuller Theory claim. Also see [Joshi, 2021b], [Joshi, 2023c].

§ 1.2.3 Myth Mochizuki's anabelian reconstruction theorem [Mochizuki, 2015, Theorem 1.9], $\Pi^{temp}_{X/E;K}$ determines the isomorphism class of X/E. Hence [Mochizuki, 2021a,b,c,d] is a problematic theory.

Sources For This Myth [Scholze and Stix, 2018], Discussions on Peter Woit's Blog [Scholze et al., April 2020].

This Myth is False Using $\Pi^{temp}_{X/E;K}$, Mochizuki's reconstruction theorem [Mochizuki, 2015, Theorem 1.9] determines the isomorphism class of X/E but it does not determine K. [Remember *Mochizuki's Key Principle of Inter-Universality* requires us to remember geometric base-points.]

§ 1.2.4 Myth [Scholze and Stix, 2018, Remark 9] is correct and hence there is no valid theory in [Mochizuki, 2021a,b,c,d].

Sources For This Myth [Scholze and Stix, 2018], Scholze's comments to my work on MathOverFlow; where Will Sawin backs this assertion by Scholze.

This Myth is False While [Scholze and Stix, 2018, Remark 9] is the central thesis of the Scholze-Stix Report, this remark is completely false. [Mochizuki's discussion of this remark appears in [Mochizuki, 2018, (C5)] (Mochizuki refers to Remark 9 as Remark 8) and my discussion here should be read with his comments and with § 1.2.4.1.] Unfortunately, despite having pointed this out explicitly in my response (on David Robert's blog), this incorrect argument has been repeated by Scholze and few others in defending [Scholze and Stix, 2018, Remark 9].

The said remark is based on an incorrect understanding of what [Mochizuki, 2015, Theorem 1.9] provides and what is actually needed in [Mochizuki, 2021a,b,c,d]. Let me explain this point. The important difference is this: using the group $\Pi^{temp}_{X/E;K}$, the said theorem ([Mochizuki, 2015, Theorem 1.9]) provides the isomorphism class of the curve X/E as asserted but it does not provide the geometric base-point information (given by the alg. closed perfectoid field K). However, Mochizuki's Key Principle of Inter-Universality requires one to remember the geometric base-point information. One clear way of saying this, via the main theorem of [Joshi, 2021a], [Joshi, 2022], is this: *[Mochizuki, 2015, Theorem 1.9] does not determine the arithmetic holomorphic structure at all; however such structures are central to [Mochizuki, 2021a,b,c,d].* This is the sense in which Mochizuki uses the cited theorem in the context of [Mochizuki, 2021a,b,c,d]. This is an important point and Mochizuki should have alerted Scholze-Stix and other readers (*myself included*) to this point with complete clarity and adequate emphasis (this would require dealing with geometric base-points the way my work does). At any rate, the conclusion of the said theorem cannot be used to arrive at the conclusion [Scholze and Stix, 2018, Remark 9] in the context of [Mochizuki, 2021a,b,c,d]. [Also see § 1.2.1.]

Note § 1.2.4.1 : This document was sent to Scholze for an early read and his comments. After detailed conversations with Scholze (June 2024), I can say that [Scholze and Stix, 2018, Remark 9] arose because of Mochizuki's emphasis, and advertisement by him and others, of the primacy of group theory (anabelian) as the *raison d'etre* of [Mochizuki, 2021a,b,c,d]. On the other hand, Scholze and I are now in agreement (June 2024) that Mochizuki's emphasis should have been on the primacy of geometric and arithmetic objects of the theory (with an adequate demonstration of the existence of such objects). As an algebraic geometer, well-versed in classical Teichmuller Theory, I have always recognized that Teichmuller Theory (of any sort: classical, p-adic, arithmetic) is about geometric objects and in my work, I have emphasized the primacy of geometric and arithmetic objects from the very beginning, with relevant groups arising as symmetries of such objects, and my reading and criticism of both [Mochizuki, 2021a,b,c,d] and [Scholze and Stix, 2018, Scholze, 2021] has been based on this viewpoint.

Note § 1.2.4.2 : This document has also been sent to Mochizuki for an early read and his comments. I am currently awaiting his response.

§ 1.2.5 Myth All Hodge-Theaters in [Mochizuki, 2021a,b,c,d] are isomorphic, hence there is no theory in [Mochizuki, 2021a,b,c,d].

Sources For This Myth [Scholze, 2021], Comments by Scholze and others on Peter Woit's blog [Scholze et al., April 2020].

This Myth is False Collection of distinct Hodge-Theaters (with a fixed decoration) is a proxy for a variation of (p-adic) Hodge Structures (at all primes p simultaneously) [Mochizuki, 2021a, Page 25] and in particular Hodge-Theaters arise from distinct geometric and arithmetic data. It is certainly true that all Hodge-Theaters (with a fixed decoration) are all isomorphic, *but* such an isomorphism comes at the expense of forgetting the arithmetic and geometric data the Hodge-Theaters arise from and doing this is analogous to asserting that all Riemann surfaces of a fixed genus are homeomorphic–a statement which is true but not the goal in the study of Riemann surfaces. So the identification of Hodge-Theaters asserted in [Scholze, 2021] serves no useful mathematical purpose as we are averaging over contributions of distinct arithmetic geometric data in [Mochizuki, 2021a,b,c,d]. [Mochizuki does not provide an adequate justification for the existence of distinct Hodge-Theaters, but my work provides the most natural way of understanding how p-adic Hodge Theory enters the picture and demonstrates clearly how distinct Hodge-Theaters arise from distinct geometric data (see [Joshi, 2024b]).]

§ 1.2.6 Myth There is no purpose to Mochizuki's weird definitions and formalism in [Mochizuki, 2021a,b,c,d].

Sources For This Myth [Scholze and Stix, 2018], Discussion on Peter Woit's blog [Scholze et al., April 2020].

This Myth is False In hindsight gained from my work, Mochizuki's formalism is an approximate proxy for modern p-adic Hodge Theory (at all primes simultaneously). This is an important point many p-adic Hodge Theorists have missed. Notably, Mochizuki's use of perfect Frobenioids (together with their realifications) in [Mochizuki, 2021a,b,c,d] is to placate the absence of machinery of p-adic Hodge theory such as tilting which was not available to Mochizuki. [Mochizuki's group theoretic approach does make this aspect extremely difficult to see but my work makes this completely transparent (see [Joshi, 2021a, 2022, 2023a]) and [Joshi, 2024b] contains a comparison of these perfect Frobenioids and perfectoids in one of the appendices.]

§ 1.2.7 Myth The theory of [Joshi, 2021a, 2022, 2023a] adds on information which serves no purpose in this discussion related to [Mochizuki, 2021a,b,c,d].

Sources For This Myth This myth may have its origins in conversations (2023) about my work on MathOverFlow and on Peter Woit's Blog.

This Myth is False As pointed out in [Joshi, 2021a, 2022, Remark 4.3], the local Arithmetic Teichmuller Space constructed in [Joshi, 2022, Definition 5.1] bears a formal similarity to the diamond $X_{L_v}^{\diamondsuit}$ associated to the analytic adic space $X_{L_v}^{ad}$ given in [Scholze, 2017, Definition 15.5]. I had already pointed out (as far back as 2022) this similarity on a number of social media sites. [Note For precise mathematical assertion see [Joshi, 2022, Remark 4.3 and Definition 5.1]. [Strictly speaking, [Joshi, 2022, Def. 5.1] deals with a slightly more general object than $X_{L_v}^{\diamondsuit}$–but this technical point may be ignored for simplicity.]

To put it clearly: in dimension one and genus one, the theory developed in [Joshi, 2021a, 2022, 2023a] so closely intertwines the theories of [Scholze, 2012, 2017], [Fargues and Fontaine, 2018], [Mochizuki, 2021a,b,c,d] that any attempt at separation based on this myth has no factual mathematical foundation.

§ 1.3 Mochizuki, Scholze-Stix claims compared The following table provides a comparison of hypotheses of Mochizuki and Joshi and the omissions of these hypothesis by Scholze-Stix leading to incorrect mathematical conclusions.

Comparison of claims of Mochizuki, Scholze-Stix and Joshi

object	Mochizuki	Scholze-Stix	Joshi
geom. base point	Central role of arbitrary geometric base-points as a proxy for deformations of arithmetic [Mochizuki, 2021a, § I3, Page 25]	ignored (see § 1.2.1, § 1.2.3)	included (in the data of an arithmeticoid) as arbitrary alg. closed perfectoid fields [Joshi, 2021a, 2022]
how is this used?	domain and codomain of all key operations refer to distinct geom. base-points [Mochizuki, 2021a, § I3, Page 25]	incorrectly identify the domains and codomains leading to incorrect conclusions	naturally show Mochizuki's requirements [Joshi, 2024b]
Valuation data	encoded in realified Frobenioids [Mochizuki, 2021a]	no mention of distinct valuation data	works with valuation data instead of Frobenioids [Joshi, 2021a, 2022, 2023a]
Why needed?	for computing local and global arith. degrees	ignored	as in Mochizuki but without using Frobenioids [Joshi, 2023a, 2022]
log-Links	log-Links aka Mochizuki's proxy for Frobenius (at each prime) [Mochizuki, 2021c]	ignored (see § 1.2.5)	works with the Frobenius morphism instead of a proxy [Joshi, 2023b,a].
Theta-Links	Central role of Theta-Links Claimed via theory of Frobenioids [Mochizuki, 2021a]	argue that such objects cannot exist (see § 1.2.5)	demonstrates the existence both at local and global level. [Joshi, 2023b,a, 2024b]
distinct Arith. Holomorphic Structures	Asserted in [Mochizuki, 2021a] but existence is not clearly established	declare that these cannot exist (see § 1.2.1, § 1.2.4)	Demonstrates the existence and deformation property via arbitrary alg. closed perfectoid fields or equivalently by using arbitrary geom. base-points [Joshi, 2021a, 2022, 2023a]

NOTE

(1) Despite the fact that Mochizuki asserts his Key Principle of Inter-Universality, there is no mention of the required input data (see the table accompanying § 1.2.1) in [Mochizuki, 2021a,b,c,d], and on two separate occasions, Mochizuki denied the relevance of alg. closed perfectoid fields to [Mochizuki, 2021a,b,c,d].

(2) One cannot build a valid theory, as [Mochizuki, 2021a,b,c,d] claims to do, by requiring arbitrary geometric base-points for tempered fundamental groups but work solely over $\bar{\mathbb{Q}}_p$. This is the reason for my mathematical objections to [Mochizuki, 2021a,b,c,d]. My work fixes this central issue and builds a robust (and even more general) theory with all properties claimed by Mochizuki for his IUTT.

§ 1.4 Conclusion

(1) The conclusion of this report is independent of the proof of the abc-conjecture.

(2) As § 1.2, § 1.3 shows, the reports [Scholze and Stix, 2018], [Scholze, 2021] omit most key mathematical assumptions and aspects of Mochizuki's Theory (see Note § 1.2.4.1). Mochizuki's report on [Scholze and Stix, 2018] is here. My work and this document shows that arguments and claims of [Scholze and Stix, 2018], [Scholze, 2021] *and* [Mochizuki, 2018], [Mochizuki, 2022] have no mathematical importance except as a matter of historical record.

(3) I think that Mochizuki has mounted an incredibly weak defense of his work, but it would be a travesty of mathematics to not acknowledge that the assessment of Mochizuki's claims was based on a poor and unsound understanding, *by many*, of the mathematical principles which have guided his claims. [This is independent of whether or not the abc-conjecture has been proven in his works.]

(4) *However, let me make this absolutely clear:* Mochizuki's unwillingness to have mathematical conversations or to recognize my work, which bring mathematical clarity to his work, continues to be a vexing issue (anabelian geometers around Mochizuki have also followed his suit on this). Moreover, I strongly protest the unprofessional behavior (e.g. name-calling and personal insults) that he has resorted to in this matter.

(5) Notably, there exists a valid Teichmuller Theory of Number Fields as Mochizuki has claimed (this is fully demonstrated in [Joshi, 2023a]).

(6) I demonstrate that Mochizuki's formalism in [Mochizuki, 2021a,b,c,d] is a proxy for modern p-adic Hodge Theory (at all primes simultaneously) and my work provides a mathematically precise way of arriving at his claims by circumventing his formalism.

(7) The ideas espoused by Mochizuki in [Mochizuki, 2021a,b,c,d] are highly non-trivial even though, he and all his 'experts,' have been less than lucid about what is it that his theory claims. [While Mochizuki may be unaware, I have heard that some of the 'experts' have studied my work to better understand his claims.]

(8) My position on whether or not Mochizuki has proved the abc-conjecture is still open (as my preprint [Joshi, 2024a] still remains under consideration). In other words, I'm currently neutral on the matter of the abc-conjecture. *However, I continue to work on [Joshi, 2024b,a] to tie up all the loose ends.* Updates of [Joshi, 2023a, 2019, 2024b] posted.

§ 1.5 Time-line of events Here is a brief time-line of events as I have witnessed them:

March 2012 Mochizuki's papers are released online and published as RIMS (Kyoto) Preprints.

Oct 2012 Vesselin Dimitrov posts his counter examples to Mochizuki's stated estimate in the March 2012 (and RIMS Preprint) version of [Mochizuki, 2021d]. Mochizuki points out corrections in this response and Dimitrov withdraws his objections in his MO post (and Mochizuki's corrections appear in [Mochizuki, 2021d]).

2015 [Fesenko, 2015] is published (see § 1.2.1).

2015 First IUTT Workshop.

2015 Brian Conrad's Report on the above conference.

2016 Second IUTT Workshop.

2016 [Mochizuki, 2016] is published and it provides insights into his proof by means of proofs of [Amorós et al., 2000, Zhang, 2001] of geometric case of Szpiro inequality. [On the other hand [Scholze and Stix, 2018], perhaps unaware of this, ignore the geometric case and miss the most important of all clues to the claims of [Mochizuki, 2021a,b,c,d].]

2016 In the context of [Mochizuki, 2016], Taylor Dupuy posts a two part video explaining the Bogomolov Zhang Proofs. [I was not aware of Taylor's videos, however I learned of the geometric case from Mochizuki, see the next point.]

Spring 2018 I visit Mochizuki for my sabbatical. Scholze and Stix visit Mochizuki in April. Mochizuki lectures to me on the Bogomolov Zhang Proofs (discussed in entry **2016** above) based on [Mochizuki, 2016] and I came to further understand the geometric case from this point of view through Mochizuki's lectures (and my independent reading of [Amorós et al., 2000, Zhang, 2001]). Because of my prior understanding of Mochizuki's work on Teichmuller Theory ([Mochizuki, 1996], [Mochizuki, 1999]), I recognized that he may be onto something highly non-trivial.

May 2018 [Scholze and Stix, 2018] is put into circulation; Mochizuki posts his report.

2019 [Joshi, 2019] (first version is posted on the arxiv).

April 2020 [Joshi, 2020a] is posted on the arxiv. [The March 2020 version had some errors in one section which were fixed in this release.] Scholze asserts that anabelomorphy (introduced in [Joshi, 2020a]) has nothing to do with Mochizuki's work; I invite Mochizuki to explain the relationship and he does by contributing to [Joshi, 2020a, 1.7], and it clearly contradicts Scholze's assertion.

April 2020 Discussion on Peter Woit's blog. Taylor Dupuy argues that Scholze-Stix have missed some key points of [Mochizuki, 2021a,b,c,d]. [Also see § 1.2.1.]

July 2020 Early versions of [Joshi, October 2020] are sent to Mochizuki and Scholze. [Scholze kindly verifies the proof, but denies its role in Mochizuki's work; perfectoid fields don't sit well with Anabelian minded Mochizuki and Hoshi and they dismiss it, contradicting Mochizuki's Key Principle of Inter-Universality [Mochizuki, 2021a, § I3, Pages 25–26] even though my results provide the clearest evidence that there is an arithmetic Teichmuller Theory of the sort Mochizuki claims. Such a theory is detailed by me in [Joshi, 2021a].]

July 2020 Early version of a 'Rosetta Stone' fragment arising in my work and establishing the connection between Mochizuki's log-Links and Fargues-Fontaine Theory was sent to Mochizuki. Mochizuki denies its relevance. [The relevant mathematics is detailed in [Joshi, 2023b] and [Joshi, 2023a].]

August 2020 I posted [Joshi, 2020b] on the arxiv.

Oct 2020 Early version of the first of my arithmetic Teichmuller Theory papers [Joshi, 2021a] is posted on the arxiv.

March 2021 Cancellation (by the organizers) of my invited talk in this June 2021 IUTT meeting.

March 2021 Mochizuki's papers [Mochizuki, 2021a,b,c,d] are published. On the eve of the publication Mochizuki posts [Mochizuki, 2022] arguing 'change of logic.' [Mochizuki does not recognize my approach ([Joshi, October 2020], [Joshi, 2022]), which provides a better way of demonstrating that his theory has many distinct objects (for natural reasons).]

Summer 2021 I give several Zoom lectures to Taylor Dupuy on my approach to Mochizuki's work.

June 2021 During this meeting, Mochizuki and Hoshi (on Zoom/Slack) deny the role of algebraically closed perfectoid fields (i.e of geometric base-point datum) contradicting Mochizuki's Key Principle of Inter-Universality.

July 2021 [Scholze, 2021] is published.

Nov 2021 [Joshi, 2021b] is posted on the arxiv.

Nov 2021 David Roberts' blogpost. [I agree with this article: Mochizuki has repeatedly cited the analogy with $\mathbb{P}^1$ in the context [Mochizuki, 2021a,b,c,d]. But using this example is completely misleading (as David also notes) and does not even remotely represent the Θ-Link in [Mochizuki, 2021a,b,c,d]. See [Joshi, 2023b, 2024b] for the precise construction.]

2022 [Saïdi, 2022] is published (contains no discussion of [Scholze and Stix, 2018, Scholze, 2021]).

Spring 2022 I give several lectures on Zoom to Minhyong Kim and Dinesh Thakur. [First draft of [Joshi, 2024b] is readied.]

Oct 2022 I post a new version of [Joshi, 2022] and discuss relationship between my results, [Mochizuki, 2021a,b,c,d], and [Scholze and Stix, 2018].

Nov 2022 Correspondence with Will Sawin (Nov 11–Nov 18) explaining my work. [It was clear that he had a limited understanding of the relevant mathematics.] Scholze (Nov 22–) argues on MathOverFlow that [Scholze and Stix, 2018, Remark 9] is still valid (see § 1.2.4). Subsequently, Will Sawin backs Scholze's assertion and dismisses my work.

Nov 2022 I respond with this mathematical explanation on David Roberts' blog and in it I clearly point out the issue of geometric base-points (which is missing in [Scholze and Stix, 2018]). No response is received from Scholze or Sawin.

March 2023 [Joshi, 2023b] is posted on the arxiv.

April 2023 In response to the [Scholze and Stix, 2018] claims about Mochizuki's Corollary 3.12, I post this essay based on [Joshi, 2023b] outlining my ideas.

May 2023 [Joshi, 2023a] is posted on the arxiv. This describes an arithmetic Teichmuller Theory of Number Fields (whose existence is asserted by Mochizuki) and I also establish the geometric case of Mochizuki's Corollary 3.12.

June 2023 I post this response to "Grouchy Expert's" comments on David Roberts' blogpost regarding my work.

July 2023 Peter Woit declares victory for Scholze-Stix in this blogpost. I post a number of comments to his blogpost to remind Woit's readers of the flaws in the Scholze-Stix report. Concerning Will Sawin's posted comments see § 1.2.4.

Jan 2024 [Joshi, 2024b] is posted on the arxiv.

March 2024 [Joshi, 2024a] is posted on the arxiv. Mochizuki posts his comments on my work. Neither Mochizuki nor Scholze publicly recognize any of my work done so far but both argue that [Joshi, 2024a] must be incorrect.

[*Work in Progress:* My work (in dimension one and genus one) so closely mirrors Mochizuki's Theory that if there are any counter examples, to [Joshi, 2024a], as Mochizuki claims in his comments, then they will also apply to [Mochizuki, 2021d]. Since Mochizuki is asserting these counter examples, it is clear to me that either he has a poor understanding of my work [Joshi, 2021a, 2023b,a, 2024b,a] or the main constructions of [Joshi, 2024b] needs some minor corrections. Unfortunately, so far, Mochizuki has not shown willingness to engage in any mathematical conversations with me, and hence it will take me some time to tie up any remaining issues.]

May-June 2024 *Work in Progress:* Updated several preprints ([Joshi, 2019], [Joshi, 2020a], [Joshi, 2023a], [Joshi, 2024b]) on the arxiv. In updates to [Joshi, 2023a] and [Joshi, 2024b] I provide additional clarifications about how I deal with local and global phenomena.

Meanwhile, Scholze and I are having a respectful and professional conversation (on going) as I work to clarify his questions; while I continue to wait for Mochizuki's response to my emails.

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