–Math 582– Applied Complex Analysis
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Brief list of topics covered

(1) A careful but possibly accelerated intro to complex variables, (2) applications of contour integration to the description of solutions to ordinary differential equations and partial differential equations, (3) special functions and asymptotic analysis, and (4) Riemann-Hilbert problems.

Description

The theory of functions of a complex variable connects many diverse topics in mathematics. These connections have been forged out of the fundamental scientic goal:

To describe, understand, and predict, with detail and precision, what happens in the system under consideration.

The 'system' mentioned above could be the solution to a nonlinear partial differential equation, or it could be something as basic as the number of primes less than a number N. The issues which are at stake in complex variables are fundamentally analytical, and have been pounded out of the molten material of these and other diverse areas.

Integrals, special functions, asymptotics, and applications

In many cases, one is interested in critical behavior when some parameter, be it a mathematical parameter, or an actual dial or knob, is stressed. An example is the following: consider the partial differential equation

$$\frac{\partial}{\partial t}u - \frac{\partial^3}{\partial x^3}u = 0, \tag{1}$$

where at $t = 0$ it is known that the solution is a smooth and rapidly decaying function. **Question:** Is it possible to provide a complete characterization of the behavior of the solution in the long-time limit? The answer is surprisingly complicated: there are regions of exponential decay, and also regions of ordered oscillations, and these regions evolve in a straightforward way as time evolves.

The first topic shall be to develop analytical techniques for the anaylsis of integrals as they arise in applications.

Riemann–Hilbert Problems

The second topic will be Riemann-Hilbert problems. This is a very modern and rapidly developing area of research in applied analysis. The area is fundamentally complex variables, but is intimately connected with a number of different mathematical arenas: combinatorics, partial differential equations, random matrix theory, and approximation theory, to name a few. We will study a few of these connections, and see how asymptotic analysis plays a central role in tying many things together. We will learn recently developed (over the last 10 years) techniques for the asymptotic analysis of solutions of Riemann-Hilbert problems.