Proposed course:

Gauge Theory and Topology

by Sergey Cherkis

Gauge theory is at the heart of modern physics and geometry. In physics, all fundamental forces of Nature are described in terms of gauge theory. In mathematics, various topological and differentiable invariants are defined using gauge theory. This course introduces bundles, connections, and curvature leading to the Yang-Mills theory. Then, it focusses on using these to construct topological classes, building up to the Chern-Weil theory.

The significance of topology will be illustrated by topological defects in superfluid helium, textures in nematic liquid crystals, instantons and magnetic monopoles. In all of these cases topological classes ensure the presence and stability of the corresponding physical objects.

Such mathematical questions as the dimension of the solution space of an (elliptic) differential equation and physics related to quantum anomalies of various symmetries are understood in terms of the index theory, which provides answers that are also topological in nature. We conclude the course by stating (without proof) the index theorem of Atiyah and Singer and study some of its applications.

Prerequisites:

Math 523A, 534A and 534B.

Texts:

Mikio Nakahara, "Geometry, topology, and physics," Taylor & Francis 2003, ISBN: 0750306068

Michael F. Atiyah, "Geometry of Yang-Mills Fields," Pisa (1979)

Roger Penrose, "On the cohomology of impossible figures," Leonardo, Vol. 25, No. 3/4, Visual Mathematics: Special Double Issue (1992), pp. 245-247 https://www.jstor.org/stable/1575844

Approximate Schedule:

- Various Homology and Cohomology Groups and their relations (2 weeks).
- Principal and vector bundles (2 weeks).
- Moduli spaces of vacua and topological defects (1 week).
- Connections on fiber bundles (1 week).
- Characteristic classes: Stiefel-Whitney, Pontrjagin, Chern classes and Chern characters (2 weeks).
- Formulation of Atiyah-Singer index theorem (with the exact statement, but, most likely, without the full analytic proof) (2-3 weeks).
- Monopoles and Instantons in Yang-Mills theory (2-3 weeks).
- Moduli Spaces of Monopoles and Instantons (time permitting)

Expected Learning Outcomes:

By the end of this course students are expected to have a good command of the Chern-Weil theory of characteristic classes, know how to apply it to evaluate their values for specific bundles, and use this knowledge to effectively use index theory to find the dimension of the space of solutions of linear equations, such as the Dirac equation, for example. (The proof of the index theorem will not be covered in this course.)

Familiarity with the notions of gauge theory solitons (in particular monopoles and instantons).

Students expected to be able to use the Atiyah-Drinfeld-Hitchin-Manin construction to explicitly produce one instanton and the Nahm construction to produce one monopole.

Ability to use topological classes to predict existence of stable solitonic defects in physical theories.