Special topics class Fall 2023: Homological Algebra, MATH 518

Prerequisites: Algebra, MATH 511A/B.

The aim of this course is to give an introduction to the objects and methods in homological algebra. The methods of homological algebra are fundamental in many different areas of mathematics such as group theory, algebraic number theory, commutative and noncommutative ring theory, representation theory, Lie algebras just to name a few. Moreover, quite a few of the conjectures in these areas are formulated in the language of homological algebra.

Historically, homological algebra originated as an offspring of algebraic topology. In algebraic topology, vector spaces (or more generally abelian groups), are attached to a topological space, and linear maps are attached to continuous maps between topological spaces. If a map is a homeomorphism then this is reflected by the corresponding linear map being a vector space isomorphism. This connection allows us for example to conclude that two topological spaces are not homeomorphic, if the vector spaces attached to them are of different dimension. Mathematicians had realized in the 1940's that the underlying formalism of attaching vector spaces to topological spaces could be extended to many algebraic structures. This is basically the topic that homological algebra is dealing with.

In the first part of the course we will study the fundamental background of homological algebra, which is the theory (and language) of categories. We will therefore start with the basics of categories, functors, and natural transformations with a special emphasis on abelian categories and additive categories. An important role is played by the notion of an equivalence of categories, which is more general than the (naive) notion of an isomorphism of categories and also much more useful. We will look at several important examples of categories such as the category of abelian groups, the category of modules over a not necessarily commutative ring, the category of modules over a Lie-algebra and more. As important functors we will look at so called Hom functors and and tensor product functors.

In the second part, we will introduce important objects in the category of modules over a ring, such as projective modules, injective modules, and flat modules. We will discuss the notion of a chain complex of modules: basically, a chain complex of modules is just a sequence of modules together with a map from one module of the sequence to the next, such that the composition of a map with the next one is zero. To a module M we will construct a special complex called a projective resolution of M. This construction will enable us to attach to the module M together with a specific Hom functor a sequence of groups called the cohomology groups $H^n(M)$ of M, similar to the case of attaching vector spaces to topological spaces. We will show that this construction is independent of the choice of the projective resolution we have chosen. Moreover, we will use the construction to study various questions: when is $H^n(M)$ zero for all n and how are $H^n(M)$ and $H^n(N)$ for two modules M and N related? Similarly, we will investigate analogous constructions for other functors such as the tensor product functors.

In the third part we will study how far the construction can be generalized to other algebraic systems. In particular, we will look at interpretations of cohomology groups in more down-to-earth terms with respect to groups, Lie-algebras, and associative algebras. It will turn out that cohomology is strongly related to the concept of extensions of groups, modules, and Lie-algebras. For the case of associative algebras, it will lead to the definition of the Brauer group, which can be related to local class field theory. We will clarify this connection and also study how cohomology groups can be determined with the aid of specialized software.

If there is time left, then the fourth part will be an introduction to more advanced topics such as the notion of a triangulated category, that of a derived category, where chain complexes are taken "seriously", Ext-algebras, and the technique of spectral sequences.

Literature:

Hilton , Stammbach: A Course in Homological Algebra, Graduate Text in Mathematics Nr 4., Springer, 1997.

Rotman: An Introduction to Homological Algebra, Academic Press, 1979.

Gelfand, Manin: Methods of Homological Algebra, Springer, 1996.

Weibel: An Introduction to Homological Algebra, Cambridge studies in advanced mathematics 38, Cambridge University Press, 1995.

MacLane: Homology, Classics In Mathematics, Springer, 1995.

Cartan, Eilenberg. Homological Algebra, Princeton University Press, 1956.