ALGEBRA QUALIFYING EXAMINATION

AUGUST 2015

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers. Say what you mean, mean what you say. Any ring denoted R is a commutative ring with identity.

1A. Let p a prime and let F be a finite field of characteristic p. Determine the conjugacy classes of elements of order p for the group $\mathrm{GL}_2(F)$ and for the group $\mathrm{SL}_2(F)$.

1B. Prove that $\det(I + XY) = \det(I + YX)$ for *any* two $n \times n$ matrices X, Y with real entries.

2A. Classify (up to isomorphism) the groups of order 2015. (Note that $2015 = 5 \times 13 \times 31$.)

2B. Show that the group $\mathrm{SL}_2(F_4)$ is isomorphic to A_5 by giving an explicit isomorphism.

3A. Let R be an infinite integral domain with finitely many units. Show that R has infinitely many distinct maximal ideals.

3B. Suppose that the commutative ring R is neither the zero ring nor a field. Prove that the polynomial ring $R[x]$ is *not* a PID.

4A. Let $f(x) = a_n x^n + \cdots + a_0 \in \mathbb{Q}[x]$ be a *palindromic polynomial*; that is, the coefficients of f satisfy $a_n = a_0$, $a_{n-1} = a_1$, and more generally $a_{n-i} = a_i$ for all $0 \leq i \leq n$. Prove that if $n > 2$ then the Galois group of f (i.e., the group $\mathrm{Gal}(K/\mathbb{Q})$ where K is a splitting field of f) is not S_n.

4B. Compute the splitting field of $(x^3 - 2)(x^2 + x + 1)$ over $\mathbb{Q}$ and determine all subfields.

5A. Let A be an abelian group with generators a, b, c, d and relations $a+b-c+d = 0$, $a - b + c + d = 0$, and $-a + b + c + d = 0$. List all homomorphisms from $B = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ to A.

5B. Suppose that A and B are $m \times m$ and $n \times n$ matrices with complex entries, respectively. Show that the linear transformation $A \otimes I_n - I_m \otimes B$ on $\mathbb{C}^m \otimes \mathbb{C}^n$ is invertible if and only if A and B have no eigenvalues in common. (Here I_m and I_n denote the $m \times m$ and $n \times n$ identity matrices, respectively.)