# ALGEBRA QUALIFYING EXAMINATION

JANUARY 2015

Do either one of $nA$ or $nB$ for $1 \leq n \leq 5$. Justify all your answers. Say what you mean, mean what you say. Any ring denoted $R$ is a commutative ring with identity.

1A. Let $A$ be an $n \times n$ matrix with entries in $\mathbb{R}$. Prove that the rank of $A$ is equal to the rank of $A^T A$.

1B. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $2 \times 2$ matrices with real entries. Define $A \otimes B$ to be the $4 \times 4$ matrix which in block form is given by $\begin{pmatrix} Ab_{11} & Ab_{12} \\ Ab_{21} & Ab_{22} \end{pmatrix}$. Prove that $A \otimes B$ is invertible if and only if both $A, B$ are invertible.

2A. Let $G$ be a group. A subgroup $H < G$ is said to be a *characteristic subgroup* if $\varphi(H) = H$ for every automorphism $\varphi$ of $G$.
(a) Prove that every characteristic subgroup is normal.
(b) Give an example of a group $G$ and a normal subgroup $H$ such that $H$ is not characteristic.

2B. Let $G$ be a group with exactly 3 conjugacy classes. Prove that either $G \simeq S_3$ (the symmetric group on 3 letters) or $G \simeq C_3$ (the cyclic group of order 3).

3A. Let $R$ be an integral domain containing a subring $k$ which is a field.
(a) If $R$ is finite-dimensional as a $k$-vector space, prove that $R$ is a field.
(b) Show by example that if $R$ is not finite-dimensional over $k$, then $R$ need not be a field.

3B. Determine, with proof, all of the ideals in the ring $\mathbb{Z}[x]/(2, x^3 - 1)$.

4A. Suppose that $K/\mathbb{Q}$ is a Galois extension with $[K : \mathbb{Q}]$ odd. If $K$ is the splitting field of the polynomial $f(x) \in \mathbb{Q}[x]$, prove that all roots of $f(x)$ are real.

4B. Give, with proof, an explicit example of a field extension $E/\mathbb{Q}$ having the following two properties.
- There are exactly two fields $K, L$ lying strictly between $\mathbb{Q}$ and $E$, and
- Neither $K \subset L$ nor $L \subset K$.

5A. Let $R$ be a commutative ring (with 1). Recall that an $R$-module $M$ is *torsion* if for every $m \in M$ there exists a nonzero $r \in R$ with $rm = 0$, and is *torsion–free* if $M$ contains no nonzero torsion submodules.
(a) If $M$ and $N$ are torsion $R$-modules, prove that $M \otimes_R N$ is also torsion.
(b) Let $R = \mathbb{C}[X, Y]$ and let $M := (X, Y)$ be the ideal of $R$ generated by $X$ and $Y$.
  (i) Show that $M$ is torsion–free as an $R$-module.
  (ii) Prove that $X \otimes Y - Y \otimes X \in M \otimes_R M$ is nonzero. **Hint:** Consider the map $M \times M \to \mathbb{C}$ given by $(f, g) \mapsto (\partial f/\partial X)(0, 0) \cdot (\partial g/\partial Y)(0, 0)$.

(iii) Prove that the $R$-submodule of $M \otimes_R M$ generated by $X \otimes Y - Y \otimes X \in M \otimes_R M$ is torsion, and conclude that $M \otimes_R M$ is *not* torsion–free.

5B. List as many non-commutative semisimple $\mathbb{R}$-algebras of $\mathbb{R}$-dimension 8 as you can. (1 point for each distinct isomorphism class of algebras in your list, to a maximum of 10.)