ANALYSIS QUALIFYING EXAM

PLEASE SHOW ALL YOUR WORK

GOOD LUCK!

PROBLEM 1

Let p and q be positive numbers. Prove that the integral

$$\int_0^1 \int_0^1 \frac{dxdy}{x^p + y^q}$$

converges if and only if $p^{-1} + q^{-1} > 1$.

PROBLEM 2

Find the following limit:

$$\lim_{n\to\infty} \int_0^\infty \frac{dx}{\left(1+\frac{x}{n}\right)^n x^{1/n}}.$$

Justify all steps.

PROBLEM 3

Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Prove that

$$\left(\int_a^b f(x)\sqrt{|\cos x|}dx\right)^4 + \left(\int_a^b f(x)\sqrt{|\sin x|}dx\right)^4 \le \left(\int_a^b |f(x)|dx\right)^4.$$

PROBLEM 4

Let $f_n(x)$ be a sequence of differentiable functions on the interval $[0,1]$. Suppose that $f_n(x) \to 0$ pointwise and that $|f_n'(x)| \le M$ for some constant M that is independent of n and x. Prove that $f_n(x) \to 0$ uniformly.

PROBLEM 5

Let S^1 be a circle of length 2π. We denote a coordinate on S^1 by θ. As usual, θ and $\theta + 2\pi$ correspond to the same point on S^1. Let $m = d\theta$ be the Lebesgue measure on S^1. Let $M \subset S^1$ be a measurable set such that $m(M) \ge 3\pi/2$. Let

$$X = \{\theta \in S^1 : m(M \cap (\theta - 0.1, \theta + 0.1)) \le 0.1\}.$$

Prove that $m(X) \le \pi$.
Hint. You may consider the intersection of the sets $\{(\omega,\theta) \in S^1 \times S^1 : |\omega - \theta| < 0.1\}$ and $M \times S^1$.

Problem 6

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Let $f \in L^2(X, \mathfrak{M}, \mu)$. Prove that there exist functions $g \in L^1(X, \mathfrak{M}, \mu)$ and $h \in L^\infty(X, \mathfrak{M}, \mu)$ such that $f = g + h$ and $||g||_1 + ||h||_\infty \leq 2||f||_2$.