REAL ANALYSIS QUALIFYING EXAM, JANUARY 2023

Please show all of your work and state any basic results from analysis which you use.

1. Suppose that ϕ is an odd smooth (i.e. C^∞) function on $\mathbb{R}$. Show that the function $\frac{\phi(x)}{x}$ can be extended to define a continuous function on $\mathbb{R}$. Is this extended function necessarily smooth?

2. a) Determine the values of a such that $f(x,y) = (1-xy)^{-a}$ is m_2-integrable on $[0,1]\times[0,1]$, where m_2 is Lebesgue measure.

b) Define

$$F(a) = \int_0^1 \int_0^1 (1-xy)^{-a} dm_2(x,y),$$

for a such that the integrand is m_2-integrable. Is F differentiable on this domain, and if so, what is its derivative?

3. Suppose that $F : [0,1] \to \mathbb{R}$. Show that there is a constant M such that $|F(x) - F(y)| \le M|x-y|$ for all $0 \le x, y \le 1$ iff F is absolutely continuous and $|F'(x)| \le M$ for Lebesgue almost everywhere x.

4. Suppose $1 \le p < \infty$ and $f \in L^p(\mathbb{R}, dx)$. Show that

$$\lim_{x\to\infty} \int_x^{x+1} f(t)dt = 0$$

5. Suppose that $f(x) = x$ on $[-1/2, 1/2)$ and extend f periodically.

(a) Find the Fourier series of f (Either the real or complex series is acceptable).

(b) Use (a) to prove Euler's theorem

$$\sum_{k=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

6. For any Lebesgue measurable function $f : [0,1] \to \mathbb{R}$ we define, for any $t \ge 0$, $\rho_f(t) = m(\{x : |f(x)| \ge t\})$.

(a) If $f \in L^p([0,1])$ for $1 \le p < \infty$, show that there is a constant C such that $\rho_f(t) \le \frac{C}{t^p}$.

(b) Give an example of a function $f \notin L^2([0,1])$ for which $\rho_f(t) \le \frac{1}{t^2}$.