# Geometry/Topology Qualifying Exam, Fall 2018

1) Let $f$ be a holomorphic function $f$ on $\mathbb{C}$. Let $L(z) = \frac{f'(z)}{f(z)}$.

a) Describe the singularities of the function $L(z)$. Hint: Recall that $f$ has a zero of order $k \in \mathbb{N}$ at $z_0$ if and only if there exists a holomorphic function $g$ on $\mathbb{C}$ such that $f(z) = (z - z_0)^k g(z)$, where $g(z_0) \neq 0$.

b) Calculate the residue of $L(z)$ at each of its singular points.

c) Prove that for a closed curve $C$ enclosing a simply connected bounded domain $R \subseteq \mathbb{C}$,

$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i N$$

where $N$ is the number number of zeroes (with multiplicity) of $f$ in the region $R$.

2) Let $M$ and $N$ be a smooth manifolds of dimension $m$ and $n$, respectively.

a) Give a definition of the tangent bundle $TM$ and the cotangent bundle $T^*M$. (Note: you do not need to show these are vector bundles, but you do need to define what vectors and covectors are.)

b) For a smooth map $f : M \to N$, give the definitions of the induced maps $f_* : TM \to TN$ and $f^* : T^*N \to T^*M$.

c) Explain why $f_*$ is not a map from smooth vector fields on $M$ to smooth vector fields on $N$, but $f^*$ is a map from smooth covector fields (or 1-forms) on $N$ to smooth covector fields on $M$.

3) Recall the stereographic projection mapping $\phi : S^2 \setminus \{N\} \to \mathbb{R}^2$, where $S^2 \subseteq \mathbb{R}^3$ is the unit 2-sphere with north pole $N = (0, 0, 1)$, given by $\phi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$ with inverse given by $\psi(X, Y) = \left(\frac{2X}{X^2+Y^2+1}, \frac{2Y}{X^2+Y^2+1}, \frac{X^2+Y^2-1}{X^2+Y^2+1}\right)$.

a) Consider the 2-form

$$\omega = \frac{e^{-\left(X^2+Y^2\right)}}{1 + X^2 + Y^2} dX \wedge dY$$

on $\mathbb{R}^2$. Show $\phi^*\omega$ extends to a smooth (global) 2-form $\Omega$ on $S^2$ that is zero at exactly one point.

b) Show that any smooth 2-form on $S^2$ that is zero at exactly one point generates the de Rham cohomology group $H^2_{dR}\left(S^2\right)$. (Note: you may use your knowledge of this group.)

4) Consider the mapping $P : \mathbb{C} \to \mathbb{C}$ given by $P(z) = z(z - 2) = z^2 - 2z$.

a) Show that $P$ restricted to $\mathbb{C} \setminus \{1\}$ is a two-sheeted covering map from $\mathbb{C} \setminus \{1\}$ to $\mathbb{C} \setminus \{-1\}$.

b) Give explicit generators of $\pi_1(\mathbb{C} \setminus \{1\}, 0)$ and $\pi_1(\mathbb{C} \setminus \{-1\}, 0)$ and use these to calculate $P_*\pi_1(\mathbb{C} \setminus \{1\}, 0)$.

c) Show that $f(z) = 2 - z$ generates the group of deck transformations.

5) Let $S^1$ be the unit circle and let $D^2$ be the closed unit disk in $\mathbb{R}^2$ whose boundary is $S^1$. For each of the following spaces $X \subseteq Y$, determine whether there is a retraction $r : Y \to X$ and give a short justification:

a) Let $X = S^1$ and let $Y = D^2 \setminus \{(0, 0)\}$, the closed unit disk in $\mathbb{R}^2$ with the origin removed.

b) Let $X = S^1$ and let $Y = D^2$.

c) Let $Y$ be the solid torus $S^1 \times D^2$ and let $X = S^1 \times \{(0, 0)\}$.

d) Let $Y$ be the solid torus $S^1 \times D^2$ and let $X = \{(1, 0)\} \times S^1$ where $S^1$ is the boundary of the disk $D^2$.

6) Suppose $M$ and $N$ are connected topological manifolds (not necessarily the same dimension). Recall the wedge product $M \vee N$ which is the disjoint union of $M$ and $N$ with one point in $M$ identified with one point in $N$. Use the Mayer-Vietoris sequence to show that the $H_k(M \vee N) \cong H_k(M) \oplus H_k(N)$ if $k \neq 0$. Be sure to justify any claims about maps in the sequence. What happens at $k = 0$?