Geometry/Topology Qualifying Exam, Fall 2022

Students should complete six problems, choosing the sixth problem (on page 2).

1) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx,$$

for real number $a \neq 0$.

2) Consider the map $F : \mathbb{R}^4 \to \mathbb{R}^2$ given by $F(x, y, z, w) = (x^4 + y^3, wz)$. Let $M = F^{-1}(1, 1)$.
 a) Show that M is a smooth manifold and give its dimension.
 b) Describe the tangent space T_pM as a subspace of $T_p\mathbb{R}^4$ for the point $p = (x, y, z, w) = (1, 0, 1, 1)$.
 c) For which values of a and b is $F^{-1}(a, b)$ not a regular submanifold of $\mathbb{R}^4$?

3) Recall the real projective plane $\mathbb{RP}^2$. It has three standard coordinate charts, given by

$$(\tilde{y}, \tilde{z}) = \phi_1(x : y : z) = \left(\frac{y}{x}, \frac{z}{x}\right) \qquad (\hat{x}, \hat{z}) = \phi_2(x : y : z) = \left(\frac{x}{y}, \frac{z}{y}\right) \qquad (\bar{x}, \bar{y}) = \phi_3(x : y : z) = \left(\frac{x}{z}, \frac{y}{z}\right)$$

where

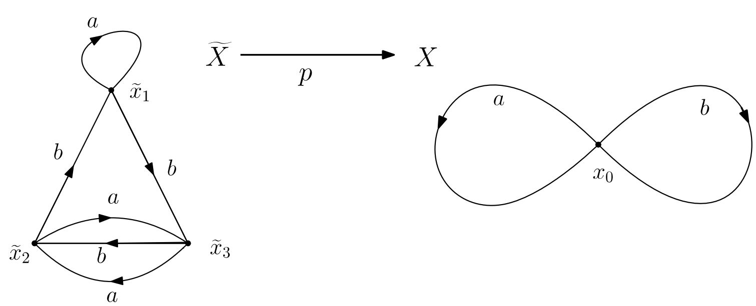
$$\phi_1 : U_1 \to \mathbb{R}^2, \qquad \phi_2 : U_2 \to \mathbb{R}^2, \qquad \phi_3 : U_3 \to \mathbb{R}^2$$

and $U_1 = \{(x : y : z) \in \mathbb{RP}^2 : x \neq 0\}$, $U_2 = \{(x : y : z) \in \mathbb{RP}^2 : y \neq 0\}$, and $U_3 = \{(x : y : z) \in \mathbb{RP}^2 : z \neq 0\}$. The inverse maps are given by:

$$\phi_1^{-1}(\tilde{y}, \tilde{z}) = (1 : \tilde{y} : \tilde{z}), \qquad \phi_2^{-1}(\hat{x}, \hat{z}) = (\hat{x} : 1 : \hat{z}), \qquad \phi_3^{-1}(\bar{x}, \bar{y}) = (\bar{x} : \bar{y} : 1)$$

Show that the covector field $(\bar{x}^2 + \bar{y}^2 + 1)^{-1} d\bar{x}$ on U_3 extends to a smooth covector field on $\mathbb{RP}^2$.

4) Consider the following covering map $p : \widetilde{X} \to X$ denoted below, where $X = S^1 \vee S^1$.

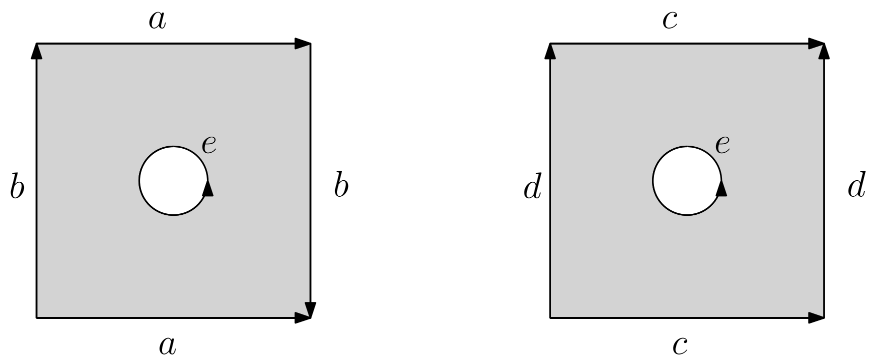


 a) Determine $p_*\pi_1\left(\widetilde{X}, \widetilde{x}_i\right)$ as a subset of $\pi_1(X, x_0)$ for $i = 1$, 2, and 3.
 b) Let $y_0 = (1, 0) \in S^1$ and let $\Phi : S^1 \to X$ be the inclusion map of the circle into the loop a such that $\Phi(y_0) = x_0$. For which value(s) of i does Φ lift to a map $\widetilde{\Phi} : S^1 \to \widetilde{X}$ such that $p \circ \widetilde{\Phi} = \Phi$ and $\widetilde{\Phi}(y_0) = \widetilde{x}_i$? Justify your answer.

5) Let ω be a smooth nonvanishing (orientation) n-form on S^n. Show that for every open subset U of S^n, there is a smooth n-form η with support in U that generates the same cohomology class as ω in $H^n_{dR}(S^n)$.

Do one of the following two problems (6A or 6B, but not both):

6A) Recall that the connected sum of closed surfaces A and B is gotten by removing a 2-disk from each of A and B and identifying the boundary circles. Let X denote the connected sum of the Klein bottle K and the torus $S^1 \times S^1$. The connected sum is indicated below with identifications a, b, c, d, e specified.



a. Give a presentation of the fundamental group of X.
b. Compute the homology groups with integer coefficients of the space X up to isomorphism.

6B) Consider the torus gotten as a surface of revolution in $\mathbb{R}^3$, and let S be the closed region encapsulated by that torus (S is called a solid torus and it is homeomorphic to $S^1 \times \overline{D^2}$, where D^2 is the open disk of dimension 2). Let T^2 denote the torus that is the boundary of S.
a. Compute the homology with integer coefficients of the quotient space S/T^2.
b. Describe the generators of the nonzero homology groups.