Geometry/Topology Qualifying Exam

January 2012

1. Compute the following integral:

$$\int_{-\infty}^{\infty} \frac{x^2}{a^4 + x^4} dx,$$

where $a > 0$ using an appropriate contour integral.

2. Suppose $f : M \to \mathbb{R}$ is a smooth function. Consider the local definition of df by

$$df = \frac{\partial f}{\partial x^i} dx^i$$

in coordinate $\left(x^1, \dots, x^n\right)$ on a neighborhood of a point $p \in M$, where we have used the Einstein summation convention.

a) Show that df is a well-defined global object by proving that if $\left(y^1, \dots, y^n\right)$ is another coordinate in a neighborhood of the point p, then

$$\frac{\partial f}{\partial x^i} dx^i = \frac{\partial f}{\partial y^j} dy^j$$

in a neighborhood of p. (Hint: you will need the transition maps going from $\left(x^1, \dots, x^n\right)$ to $\left(y^1, \dots, y^n\right)$ in order to do this calculation.)

b) Show that the naive definition of the Hessian (second derivative) given locally by

$$\frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \otimes dx^j$$

in coordinate $\left(x^1, \dots, x^n\right)$ on a neighborhood of a point $p \in M$ does not, in general, give a well-defined global object (i.e., it does not satisfy an equality as above when one changes coordinates). In addition, show that it is well-defined at a point p where $df_p = 0$.

3. For this problem, let X be a simply-connected topological space.

a) Show that any continuous map $\phi : X \to S^2$ that is not surjective is homotopic to a constant map. In the special case $X = S^1$, find an explicit homotopy between a constant map and the mapping $\psi : S^1 \to S^2$ that maps the circle to the equator of S^2 in $\mathbb{R}^3$, i.e., $\psi(x, y) = (x, y, 0)$.

b) Show that any continuous map $f : X \to S^1$ is homotopic to a constant map. In the special case $X = \mathbb{R}$, find an explicit homotopy between a constant map and the universal covering map $g : \mathbb{R} \to S^1$ given by $g(t) = (\cos t, \sin t)$.

4. a) Use Van Kampen's theorem to show that the fundamental group of the torus $S^1 \times S^1$ is presented by

$$\langle a, b : ab = ba \rangle$$

using the model of the torus given by a square with opposite sides identified. Take the basepoint to be near the boundary of the square.

b) Recall that the surface Σ_g of genus g can be obtained by taking the connected sum of g tori. Show the surface of genus two has fundamental group presented by

$$\left\langle a_1, b_1, a_2, b_2 : a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = 1 \right\rangle$$

by using Van Kampen's theorem and decomposing the Σ_2 as a connected sum of two tori. Be sure to choose an appropriate basepoint.

5. a) Suppose ω is a smooth, exact k-form. Show that $\omega \wedge \omega$ is an exact $(2k)$-form.

b) Suppose ω is a smooth, closed 2-form on S^4. Show that $\omega \wedge \omega$ vanishes somewhere.

6. The suspension ΣX of a space X is defined by taking the space $X \times [0, 1]$ and identifying all the points of $X \times \{0\}$ together and all the points of $X \times \{1\}$ together. For instance, the suspension of the circle consists of two cones joined at their base circles (homeomorphic to the 2-sphere).

Let $\pi : X \times [0, 1] \to \Sigma X$ be the quotient projection. Here are some facts about ΣX that you may use in the following problem:

A) $\pi(X \times \{1/2\})$ is homeomorphic to X.

B) $\pi(X \times (0, 1))$ is homeomorphic to $X \times (0, 1)$, where $(0, 1)$ is the open interval.

C) Let X_0 be the point in ΣX corresponding to $X \times \{0\}$ and X_1 be the point in ΣX corresponding to $X \times \{1\}$, i.e., $\pi(X \times \{0\}) = \{X_0\}$ and $\pi(X \times \{1\}) = \{X_1\}$. Then $\Sigma X \setminus \{X_0\}$ and $\Sigma X \setminus \{X_1\}$ are homotopy equivalent to the single point spaces $\{X_1\}$ and $\{X_0\}$ respectively.

Use the Mayer-Vietoris sequence to show that the reduced homology groups satisfy $\tilde{H}_{i+1}(\Sigma X) \cong \tilde{H}_i(X)$ for all $i \geq 0$ and $\tilde{H}_0(\Sigma X) = 0$. (If you are not familiar with reduced homology, use Mayer-Vietoris to compute the homology groups of ΣX in terms of the homology groups of X.)