GEOMETRY/TOPOLOGY QUALIFYING EXAM, JANUARY 2022

1. Suppose that p is a positive real number. Evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{x}{1+x^4} e^{-ipx} dx$$

2. (a) For which values of c is

$$X_c = \{(x, y, z) \in \mathbb{R}^3 : z^2 + xy = c\}$$

a smooth manifold?

(b) For the values $c = -1$, $c = 0$ and $c = 1$ answer the following questions. What is the homotopy type of X_c (e.g. is it homotopic to a point, a sphere,...)? What are the fundamental group and homology groups? Briefly explain your answers - pictures are encouraged.

3. Let $\iota : S^3 \to \mathbb{R}^4$ be the inclusion map, and consider the 3-form on $\mathbb{R}^4$ given by

$$\begin{aligned} \alpha = i_E dV = & x_1 dx_2 \wedge dx_3 \wedge dx_4 - x_2 dx_1 \wedge dx_3 \wedge dx_4 \\ & + x_3 dx_1 \wedge dx_2 \wedge dx_4 - x_4 dx_1 \wedge dx_2 \wedge dx_3 \end{aligned}$$

where E denotes the 'Euler vector field', i.e. $E|_x = x$.

(a) Evaluate $\int_{S^3} \iota^* \alpha$. **you may need the volume for S^3; it is $\frac{8}{3}\pi^2$**

(b) Let γ be the following 3-form on $\mathbb{R}^4 \setminus \{0\}$:

$$\gamma = \frac{\alpha}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^k}$$

for $k \in \mathbb{R}$. Determine the values of k for which γ is closed and those for which it is exact.

4. (a) Explain why the map

$$f : \mathbb{C} \setminus \{0, \pm 1, \pm i\} \to \mathbb{C} \setminus \{0, 1\} : z \to w = z^4$$

is a normal (or regular) covering.

(b) Find a set of generators for the subgroup of $\pi_1(\mathbb{C} \setminus \{0, 1\}, 1/2)$ which corresponds to this covering.

5. A framed knot is an embedding of $D^2 \times S^1$ into S^3, where the image of the embedding is denoted K and D^2 denotes the open two dimensional disk. Show that the 1st homology group of the knot complement, $H_1(S^3 \backslash K) \cong \mathbb{Z}$.

6. For each of the following statements, either briefly explain why the statement is true, or give a counterexample.

(a) Every exact $k-$form on a compact orientable k-dimensional manifold vanishes at some point.

(b) If X is vector field on a manifold M and $X(q) \neq 0$, then there exists a coordinate system $x_1, .., x_n$ near q such that $X = \frac{\partial}{\partial x_1}$.

(c) There exists a compact two-manifold X with $H_1(X, \mathbb{Z}) \neq 0$ and $H^1_{DR}(X, \mathbb{R}) = 0$, where H^*_{DR} denotes DeRham cohomology;