ALGEBRA QUALIFYING EXAMINATION

AUGUST 2019

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A. Let A and B be the following rational matrices: $A = \begin{pmatrix} 0 & 2 & -3 \\ -1 & 3 & -3 \\ 1 & -2 & a \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 & 0 \\ 0 & b & 0 \\ 0 & 3 & 1 \end{pmatrix}$ with $a, b \in \mathbb{Q}$.

(1) Suppose A and B are similar. Determine a and b.

(2) For that values of a and b obtained in (1), find a rational, invertible matrix P so that $P^{-1}AP$ is in Jordan canonical form.

1B. Let V be a Euclidean space over $\mathbb{R}$. Let $T : V \to V$ be a linear transform and T^* be its adjoint. Assume that $TT^* = T^*T$. Show that if $\alpha, \beta \in V$ with $T(\alpha)+\alpha = T(\beta) = 0$, then $\alpha \perp \beta$.

2A. Let G be a group generated by two elements $a, b \in G$ with $a^2 = b^2 = 1$. Show that the commutator subgroup of G is cyclic.

2B. Prove that a group of order 99 is abelian.

3A.

a) Let I be a finite integral domain. Show that I is a field.

b) Let R be a commutative ring with identity and let I be a prime ideal of R of finite index. Show that P is a maximal ideal.

3B. Let A be a commutative ring with identity. Let $I_1, \ldots, I_s$ be ideals of A such that $I_1 \cap \cdots \cap I_s = (0)$. If A/I_i is Noetherian for all $i = 1, \ldots, s$, show that A is Noetherian.

4A. Determine the Galois group for the polynomial $x^9 - 1$ over $\mathbb{Q}$ and over the field $\mathbb{F}_7$ with 7 elements and determine all subfields of a splitting field in both cases.

4B. Let $a \in \mathbb{Q}$. Assume that $f(x) = x^3 - a$ is irreducible over $\mathbb{Q}$. Determine the Galois group of $f(x)$.

5A. (1) Let R be a PID and M be a finitely generated module over R. Show that M is free if and only if M is torsion free.
(2) Let R be an integral domain. Show that an ideal I of R is a free R module if and only if it is principal. Use this to give a torsion free module over $\mathbb{C}[X, Y]$ that is not free.

5B. Let A be an abelian group generated by three elements $a, b, c \in A$ with $12a - 3b + 6c = -6a + 3b - 6c = 0$. Write the following abelian groups as a direct sum of cyclic groups: A, $A \otimes_{\mathbb{Z}} A$ and the group $\mathrm{Hom}(A, A)$ of group homomorphisms from A to A.