ALGEBRA QUALIFYING EXAMINATION

AUGUST 2021

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A. Let A and B be two nilpotent $n \times n$ matrices with entries in $\mathbb{C}$.

- a.) If $n \leq 6$ and A and B have the same minimal polynomial and the same rank, prove that A and B are similar.
- b.) Give an example of two nilpotent 7×7 matrices with the same minimal polynomial and rank which are not similar.

1B. Give a concrete example of a real matrix A with $A^5 = I$ and A not diagonalizable over $\mathbb{R}$. Show that A is diagonalizable over $\mathbb{C}$.

2A. Let G be a simple group of order 168.

- a.) If H is a proper subgroup of G, prove that $[G : H] \geq 7$.
- b.) Prove that G has no elements of order 21. **Hint:** If $g \in G$ has order 21, then $P := \langle g^7 \rangle$ is a Sylow 3-subgroup of G with $g \in N_G(P)$.

2B. Suppose that G is a finite group that acts faithfully and transitively on a finite set S. If $G_a := \mathrm{Stab}_G(a)$ for $a \in S$ show that there does not exist a nontrivial $N \lhd G$ with $N \leq G_a$.

3A. Let R be an integral domain with field of fractions F. We say that R is a *valuation ring* if for every nonzero $x \in F$, either $x \in R$ or $x^{-1} \in R$.

- a.) If R is a valuation ring and I and J are ideals in R, prove that either $I \subseteq J$ or $J \subseteq I$. Thus, the ideals in a valuation ring are totally ordered.
- b.) Conversely, prove that if the ideals of R are totally ordered by inclusion as in a.), then R is a valuation ring.

3B. Show that any principal ideal of $\mathbb{Z}[X]$ is not a maximal ideal.

4A. Let K be the splitting field of $x^{20} - 1$ over $\mathbb{Q}$.

- a.) Determine $\mathrm{Gal}(K/\mathbb{Q})$, and prove that your answer is correct.
- b.) Determine with proof the number of distinct subfields $E \subseteq K$ containing $\mathbb{Q}$, for each possible degree $[E : \mathbb{Q}]$.

4B. Show that the polynomial $2x^5 - 10x + 5 \in \mathbb{Q}[x]$ is not solvable by radicals.

5A. Let A be the abelian group with presentation

$$A := \langle x, y, z \ : \ 4x + 7y + 3z,\ 7x - 8y + 6z,\ -7x + 20y - 6z \rangle.$$

Express A as a direct product of cyclic groups of prime power order.

5B. Determine all noncommutative, semisimple rings with 144 elements up to isomorphism.