ALGEBRA QUALIFYING EXAMINATION

AUGUST 2022

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A. Determine representatives for all the conjugacy classes of elements of order dividing 8 in $\mathrm{GL}_4(\mathbb{F}_2)$ and give the orders of the representatives.

1B. Let K be field, V be a finite dimensional K-vector space and let $T \colon V \to V$ be a K-linear map. Prove there exists a positive integer n such that $V = \ker T^n \oplus \text{Image } T^n$.

2A.

a) Let G be a nonabelian, simple subgroup of the symmetric group S_n, $n \geq 5$. Show that $G \leq A_n$.

b) Show that there is no nonabelian, simple group of order 120. You are allowed to use the statement of part a).

2B. Let G be a finite group, H a normal subgroup of G, and P a Sylow p-subgroup of G. Prove that $P \cap H$ is a Sylow p-subgroup of H.

3A. Prove that if R, S are commutative rings with unity and $f \colon R \to S$ is a ring homomorphism with $f(1_R) = 1_S$ then $f^{-1}(I)$ for a prime ideal I in S is a prime ideal in R. Give an example of such a ring homomorphism $f \colon R \to S$ and a maximal ideal $I \leq S$ such that $f^{-1}(I)$ is not a maximal ideal in R.

3B. Let R be a commutative ring with unity. The *Jacobson Radical* $J(R)$ is defined as the intersection of all maximal ideals of R. Let $r \in R$. Prove that $r \in J(R)$ if and only if $1 - ry$ is a unit for every $y \in R$.

4A. Let $f(x) = x^{13} - 1$ in $\mathbb{F}_5[x]$. Determine the degree of a splitting field K for $f(x)$ over $\mathbb{F}_5$. How many elements $a \in K$ have the property $\mathbb{F}_5(a) = K$ and how many elements generate the multiplicative group of K?

4B. Let F be a field, $f(x) \in F[x]$ a polynomial of degree a prime p, and denote by K the splitting field of $f(x)$. Suppose $[K : F] = mp$ for some positive integer m. Prove that $f(x)$ is irreducible over F. Next, prove that if $m > 1$ then K is a separable extension of F.

5A. Let $A = \mathbb{Z}^n$. Suppose B is a subgroup of A generated by strictly fewer than n elements. Prove that the index $[A : B]$ is infinite. Next, for m a positive integer, prove that there are only finitely many subgroups of A with index m.

5B. Let G be a group given by the presentation $\langle a, b, c | a^2 = b^3 = c^4 = abc = 1 \rangle$. Describe the structure of the commutator factor group $G/[G, G]$ as a direct product of cyclic groups and determine the number of homomorphisms of G to $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_4$.