ALGEBRA QUALIFYING EXAMINATION

AUGUST 2023

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A. Let V be a finite dimensional vector space over a field k, and let T be a linear map on V. Suppose $f \in k[x]$ is a polynomial such that $f(T) = 0$. Suppose further that f factors over k, $f = h_1 \cdot h_2$ with $h_1, h_2 \in k[x]$ relatively prime. Set $W_1 = \ker h_1(T)$ and $W_2 = \ker h_2(T)$. Prove that W_1 and W_2 are T-invariant subspaces of V, and $V = W_1 \oplus W_2$.

1B. Let V be a finite-dimensional complex vector space, and $A\colon V \to V$ be a linear map such that $A^2 = A$. Prove that there is a basis in V in which the matrix of A is diagonal with diagonal entries 0 or 1.

2A. Show that if G is a group of order $2 \cdot 3 \cdot 5 \cdot 67$, then G is solvable.

2B. Prove that there are no simple groups of order $11 \cdot 2^n$, where $n \in \mathbb{N}$.

3A. Suppose E is a finite Galois extension field of F, and suppose $F < K < E$ with K/F Galois. Suppose $|\mathrm{Gal}(E/K)| = 5$ and $\mathrm{Gal}(K/F) \cong S_3$. Prove that $|\mathrm{Gal}(E/F)| = 30$. Further, prove there exists a Galois extension $F < L < E$ with $[L : F] = 2$, and three fields W such that $F < W < E$, each not Galois, and $[W : F] = 3$.

3B. Find the Galois group over $\mathbb{Q}$ of the polynomial $27x^3 - 63x + 7$.

4A. Let E be a field with 81 elements. Give all the subfields, how many elements $\alpha \in E$ satisfy $E = \mathbb{F}_3(\alpha)$, and determine the number of elements in E that generate E^* as a group.

4B. Let $R = \mathbb{C}[x, y]/(xy - 1)$. Find all units in R.

5A. Prove the following.

(1) Let R be a ring with unity. Prove that the set of $n \times n$ matrices $M_n(R)$ with entries in R is noncommutative when $n \geq 2$.

(2) Suppose R is a semisimple commutative ring with unity. Show that R is a direct sum of fields.

5B. Set

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $R = \mathbb{R}[x]$ and $M = \mathbb{R}^4$. Define an R-module structure on M by setting $p(x)v = p(A)v$. Find the invariant factors of M, i.e. elements $d_1, \dots, d_n \in R$ such that $d_1 \mid \cdots \mid d_n$, and $M \cong \bigoplus_{i=1}^{n} R \,/\, (d_i)$.