ALGEBRA QUALIFYING EXAMINATION

JANUARY 2019

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A. Let A be a complex 6 by 6 matrix. Suppose that $A^3 = I$. List the possible Jordan canonical forms for A.

1B. Find a real orthogonal 2 by 2 matrix P such that $P^{-1}AP$ is diagonal for

$$A = \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}.$$

2A. Let G be a finite group. Let p be the smallest prime dividing the order of G. Show that any subgroup of G of index p is normal.

2B. Show that the group given by the presentation $< a, b | a^2, b^2 >$ is an infinite group.

3A. Recall that a ring A is *artinian* if every descending chain of ideals stabilizes. Let A be commutative ring with 1 that is artinian and an integral domain. Prove that A is a field.

3B. Find a maximal ideal in $\mathbb{C}[x, y]$ that does not contain xy and find a prime ideal that is not maximal and does not contain xy.

4A. Let k be a field. Answer *true* or *false* for the following statements. If true, then very briefly sketch a proof outline (1-3 lines). If false, then state an explicit counterexample.

(1) Every field extension of k of degree 2 is normal.
(2) Every field extension of k of degree 2 is of the form $k(\sqrt{\beta})$, where $\beta \in k$.

4B. Determine the Galois group of the polynomial $x^3 - 2$

a) over $\mathbb{Q}$,

b) over $\mathbb{F}_7$,

c) over $\mathbb{F}_9$.

5A. Let R be a commutative ring with 1. Let M and N be finitely generated R-modules. Prove that the tensor product $M \otimes_R N$ is a finitely generated R-module.

5B. Let R be a finite semisimple ring with 1. Suppose that no fourth power n^4 for $n \in \mathbb{N}, n > 1$, divides $|R|$. Show that R is commutative.