ALGEBRA QUALIFYING EXAMINATION

JANUARY 2022

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A. An 11×11 matrix over $\mathbb{C}$ satisfies $A^2 = 0$. Determine the largest possible rank that such a matrix can have, and give an explicit example illustrating that this maximal rank occurrs.

1B. Let V be a finite dimensional $\mathbb{C}$-vector space and let T, U be linear maps from V to V. Show that if $TU = UT$ then T and U have a common eigenvector.

2A. Let G be a simple group of order $17971200 = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$ and H a proper subgroup of G. Prove that $[G : H] \geq 14$.

2B. Let G be a group of order 24 and assume no Sylow subgroup of G is a normal subgroup of G. Show that G is isomorphic to S_4.

3A. Let R be a commutative ring of finite cardinality, and $I_1, \ldots, I_k$ proper ideals of R that are pairwise comaximal (*i.e.* $I_j + I_k = (1)$ for all $j \neq k$). If p is the smallest prime dividing $|R|$, prove that $|R| \geq p^k$. **Hint:** Consider the quotient $R/(I_1 I_2 \cdots I_k)$.

3B. Show that for any prime p congruent to 1 modulo 4 the ring $\mathbb{Z}[\sqrt{p}]$ is not a unique factorization domain.

4A. Let $f \in \mathbb{Q}[X]$ be an irreducible polynomial of prime degree $p = \deg(f)$ with splitting field K over $\mathbb{Q}$. If $\alpha \neq \beta$ are roots of f in K with $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$, prove that $K = \mathbb{Q}(\alpha)$, and that $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/p\mathbb{Z}$.

4B. Let K be the splitting field of the polynomial $X^4 + 1$ over $\mathbb{Q}$. Compute the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

5A. Let G be the abelian group with generators x, y, z subject to the relations

$$-36x + 8y - 50z = 18x - 4y + 28z = 36x - 6y + 48z = 0.$$

Express G as a direct product of cyclic groups of prime power order.

5B. Let A be a finite dimensional, semisimple $\mathbb{C}$-algebra and let M be a finitely generated A-module. Prove that M has only finitely many A-submodules if and only if M is a direct sum of pairwise nonisomorphic, simple A-modules.