ALGEBRA QUALIFYING EXAMINATION

JANUARY 2023

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A. A complex $n \times n$ matrix A is called normal, if $\bar{A}^t A = A\bar{A}^t$. Let $A_1, \dots, A_k$ be complex, normal $n \times n$ matrices such that $A_i A_j = A_j A_j$ for all $i, j = 1, \dots, k$. Show that there exists a unitary $n \times n$ matrix U such that for all $i = 1, \dots, k$ the matrix $U^{-1} A_i U$ is a diagonal matrix.

1B. Let $A \in \mathrm{GL}_n(\mathbb{C})$, and suppose A has finite order. Prove A is diagonalizable.

2A.

(1) Show that a group of order $2^n \cdot 5$ for $n \in \mathbb{N}$ is solvable.
(2) Give an example of a non-nilpotent group of order 72.

2B. Let G be a finite group and p a prime number. Let H be the intersection of all Sylow p-subgroups of G. Prove that H is normal in G. Further, if N is any normal p-subgroup of G, prove that N is a subgroup of H.

3A.
Show that the ring $\mathbb{Z}[\sqrt{5}]$ is not a principal ideal domain.

3B. Let $\mathbb{F}_3$ denote the finite field with three elements. Let $K = \mathbb{F}_3(\alpha)$, where α is a root of $x^2 + 1$.

(1) Find a generator β of the multiplicative group of K and describe it in terms of the $\mathbb{F}_3$-basis $\{1, \alpha\}$ of K.
(2) Show that $x^4 + 1$ splits in K by writing its roots in terms of β.

4A. Determine a field K containing $\mathbb{Q}$ such that the Galois group of K over $\mathbb{Q}$ is cyclic of order 3.

4B. Let $f \in \mathbb{Q}[x]$ be an odd degree, irreducible polynomial with abelian Galois group. Prove that all the roots of f are real.

5A. Determine all semisimple rings of size 1296 up to isomorphism. How many are commutative?

5B. Suppose

$$0 \longrightarrow N_1 \longrightarrow M \longrightarrow N_2 \longrightarrow 0$$

is an exact sequence of R-modules. Prove that if N_1 and N_2 are finitely generated then M is finitely generated. Give a counterexample to the converse; explicitly describe the ring R and modules involved in your example.