

REAL ANALYSIS QUALIFYING EXAM, AUGUST 2022

Please show all of your work and state any basic results from analysis which you use.

1. For subsets A and B of \mathbb{R}^2 , define $A + B = \{x + y | x \in A \text{ and } y \in B\}$. Prove the following statements:

(a) If A is closed and B is open in \mathbb{R}^2 , then $A + B$ is open.

(b) If A is closed and B is compact, then $A + B$ is closed.

2. Define $F(\lambda) = \int_1^\infty \frac{e^{-\lambda t}}{t} dt$ for all $\lambda > 0$. Show that, for all $0 < \alpha \leq 1$, there is a constant $C_\alpha < \infty$ such that $F(\lambda) \leq C_\alpha \lambda^{-\alpha}$ for all $\lambda > 0$, but there is no $C < \infty$ such that $F(\lambda) \leq C$ for all $\lambda > 0$.

3. Suppose that $F(x)$ is a right continuous function of bounded variation on \mathbb{R} , μ is the corresponding complex measure, and $\phi(x)$ is a smooth function on \mathbb{R} having compact support. Show that

$$-\int_{\mathbb{R}} \phi'(x)F(x)dx = \int_{\mathbb{R}} \phi(y)d\mu(y)$$

Under what conditions is

$$-\int_{\mathbb{R}} \phi'(x)F(x)dx = \int_{\mathbb{R}} \phi(y)F'(y)dy$$

Clarifications: μ and F are related by $\mu((-\infty, b]) = F(b)$. In the second displayed line, $F'(y)$ denotes the pointwise derivative (as in calculus), which is known to exist for Lebesgue almost every y .

4. Determine whether the following statements are true or false and justify your answer (a picture and brief explanation is acceptable).

(a) $C(\mathbb{R}) \cap L^1(\mathbb{R}, dx) \subset C_0(\mathbb{R})$, i.e. a continuous Lebesgue integrable function vanishes at infinity.

(b) $L^1 \cap L^\infty(\mathbb{R}, dx) \subset L^2(\mathbb{R}, dx)$.

5. Let $\mathcal{P} \subset C([0, 1])$ denotes the subspace of polynomials. Determine whether the following linear functionals have continuous extensions to $C([0, 1])$:

(a) $\Phi(p) = a_0$

(b) $\Psi(p) = a_0 + a_1$

where $p(x) = a_0 + a_1x + \dots + a_nx^n$.

6. Show that the Fourier transform of a finite measure is a uniformly continuous function on \mathbb{R}^n (with the Euclidean metric).