REAL ANALYSIS QUALIFYING EXAM, AUGUST 2023

Please show all of your work and state any basic results from analysis which you use.

1. A zero of a continuous function f is *isolated* if there exists an open set containing this zero, but no other zeros of f.

(a) Show that there exist continuous functions $f : (0, 1) \to \mathbb{R}$ with infinitely many isolated zeros.

(b) If $f : [0, 1] \to \mathbb{R}$ is continuous and all of its zeros are isolated, show that f has only finitely many zeros on $[0, 1]$.

2. Find the following limit and justify your reasoning

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n \log\left(2 + \cos\left(\frac{x}{n}\right)\right)\, dx.$$

3. In what follows, let P_n denote the set of all polynomials in one variable with degree less than or equal to n, and $\mathcal{P} = \bigcup_{n\geq 0} P_n$ denote the set of all such polynomials. Also, let $\|f\|_u$ denote the uniform norm on $C([0, 1])$, i.e $\|f\|_u = \sup_{x\in[0,1]} |f(x)|$.

(a) Show that the mapping T defined by $Tf(x) = f(x) - f'(x)$ maps $\mathcal{P}$ surjectively onto itself. (Hint: If g is in P_n, what is T applied to $g + g' + g'' + \cdots + g^{(n)}$.)

(b) Show that $T^{-1} : (\mathcal{P}, \|\cdot\|_u) \to (\mathcal{P}, \|\cdot\|_u)$ is unbounded.

4. Suppose $1 \leq p < \infty$ and $f \in L^p(\mathbb{R})$. Show that

$$\lim_{y\to\infty} \int_y^\infty f(t)e^{y-t}dt = 0$$

5. We define a sequence of continuous functions on the torus $\mathbb{T}^2$ by

$$g_k(x, y) = \sum_{n=0}^{k} \sum_{m=0}^{k} \frac{e^{2\pi i(mx+ny)}}{1 + m + n^2}$$

Prove or disprove the following assertions:

(a) The sequence g_k converges uniformly on $\mathbb{T}^2$.

(b) The sequence g_k converges in $L^2(\mathbb{T}^2)$.

6. Suppose that $f \in L^1(\mathbb{R})$. Prove that for almost every $x \in [0, 1]$

$$\lim_{n\to\infty} f(x + n) = 0.$$