REAL ANALYSIS QUALIFYING EXAM, JANUARY 2024

Please show all of your work and state any basic results from analysis which you use.

1. Let (X, d) be a compact metric space and let $Y = X \times X$ be the product space equipped with the product metric $\rho((x_1, x_2), (z_1, z_2)) = d(x_1, z_1) + d(x_2, z_2)$.
(a) Show that (Y, ρ) is a compact metric space.
(b) Show that the diagonal $D = \{(x, x) : x \in X\}$ is a compact subset of (Y, ρ).

2. Let m be the Lebesgue measure and suppose that $f : \mathbb{R} \to \mathbb{R}$ is in $L^p(m)$ for some $p \geq 1$. If $\int_0^x f(y)dy = 0$ for all x, show that $f = 0$ almost everywhere.

3. Show that $f(x) = \dfrac{1 - \cos(\pi x)}{x} e^{-x}$ is in $L^1([0, \infty))$ and prove the identity

$$\int_0^\infty \frac{1 - \cos(\pi x)}{x} e^{-x}\, dx = \frac{\log(1 + \pi^2)}{2}$$

Hint: For $y \in \mathbb{R}$, we have $\displaystyle\int_0^\infty e^{-x} \sin(xy)\, dx = \frac{y}{1 + y^2}$.

4. Let x, y be distinct elements of a Hilbert space $\mathcal{H}$, and let $z \in \mathcal{H}$. Show that,

$$\left\| z - \frac{x + y}{2} \right\| < \max(\|x - z\|, \|y - z\|).$$

Using this or otherwise, show that for any given $x \in \mathbb{R}^n$ and $1 \leq p < \infty$, there is a unique y in the the set $\overline{B} = \{x : \sum_i |x_i|^p \leq 1\}$ that is the closest to x in the *Euclidean* metric, i.e. y is the unique minimizer in $\overline{B}$ for

$$F(w) = \sqrt{(x_1 - w_1)^2 + \cdots + (x_n - w_n)^2}.$$

5. In what follows M will denote a proper (linear) subspace of a normed vector space V.
(a) Give an example of a normed vector space V, a proper subspace $M \subset V$ and a sequence $x_n \in M$ such that $x_n \to y$ in V but $y \notin M$.
(b) Show that, if V is complete and M is closed, then there is a point $x \in V$ such that $\|x\| = 1$ and $\|x - y\| \geq \frac{1}{2}$ for all $y \in \mathcal{M}$.

6. Let f_n be a sequence of monotone non-decreasing functions from the interval $[a, b]$ to $\mathbb{R}$. In other words, for all $a \leq x < y \leq b$ and all $n \in \mathbb{N}$, we have $f_n(x) \leq f_n(y)$. Assume that the sequence f_n converges pointwise to a continuous function f, i.e. $f_n(x) \to f(x)$ *for every* $x \in [a, b]$. Show that the sequence f_n has to converge uniformly to f.