

GEOMETRY/TOPOLOGY QUALIFYING EXAM, AUGUST 2021

1. For each of the items below, give an example with justification or explain why no example exists:

(a) a nonconstant holomorphic function $f : \mathbb{C} \rightarrow \mathbb{C}$ which is doubly periodic, i.e. $f(z + n + im) = f(z)$ for all $z \in \mathbb{C}$ and $n, m \in \mathbb{Z}$;

(b) a meromorphic function g on \mathbb{C} having Taylor series centered at 0 with radius of convergence exactly 3;

(c) a meromorphic function h on \mathbb{C} without poles on $S^1 = \{z : |z| = 1\}$, such that $\int_{S^1} f(z)dz \neq 0$ (where the circle has the counterclockwise orientation).

2. Consider the function

$$F : \mathbb{R}^3 \rightarrow \mathbb{R}^2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y^2 - xz^2 \end{pmatrix}$$

(a) Determine the sets of critical points and critical values of F .

(b) Because the domain of F is not compact, the topology of the level sets $F^{-1}\left(\begin{pmatrix} u \\ v \end{pmatrix}\right)$ can change as $\begin{pmatrix} u \\ v \end{pmatrix}$ varies in a connected component of the set of regular values. Verify this in this example by determining the homotopy type of the level sets corresponding to regular values of the form $\begin{pmatrix} u \\ 1 \end{pmatrix}$.

3. Let $\mathbb{R} \cup \{\infty\}$ be the one point compactification of \mathbb{R} , where a coordinate for a neighborhood of ∞ is given by $y = 1/x$. Consider the vector field in \mathbb{R} given by $v = x(x - 1)\frac{\partial}{\partial x}$.

(a) Show that v extends smoothly to a vector field on $\mathbb{R} \cup \{\infty\}$.

(b) Qualitatively indicate what the flow for v looks like (i.e. indicate via a picture the important features of the flow, but you do not need to explicitly solve for the flow).

4. Consider the surface X of \mathbb{R}^3 which is defined in cylindrical coordinates by the equation

$$(r - 2)^2 + z^2 = 1$$

[Recall that in cylindrical coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = z$].

(a) Find an explicit basis for the homology (in all degrees).

(b) By introducing coordinates or parameterizing the surface, find explicit expressions for forms which represent the corresponding dual basis in DeRham cohomology.

5. Suppose that $n > 1$. Find the possible degrees (with explanation) for the following:

(a) A continuous map $f : S^n \rightarrow S^1 \times \dots \times S^1$ (the n -torus), where the sphere and n -torus have the usual orientations.

(b) A covering space map $\Sigma_4 \rightarrow \Sigma_2$, where Σ_g denotes a compact oriented surface of genus g .

6. (a) Calculate the fundamental group of the space obtained by deleting a point from the 3-torus $S^1 \times S^1 \times S^1$.

(b) Calculate the homology groups of the space obtained by deleting a point from the 3-torus $S^1 \times S^1 \times S^1$.