GEOMETRY/TOPOLOGY QUALIFYING EXAM

AUGUST 2023

Please show all of your work. GOOD LUCK!

(1) Find a conformal mapping from the first quadrant $\{z = x + iy \in \mathbb{C} : x > 0,\ y > 0\}$ to the disk $\{z :\ |z| < 1\}$.

(2) Let the mapping $F : \mathbb{R}^4 \to \mathbb{R}^2$ be given by the formulas $F(x^1, x^2, x^3, x^4) = (x^1x^2x^3x^4,\ x^1 + x^2 + x^3 + x^4)$.
a) Prove that $F^{-1}(1, 1)$ is a smooth submanifold of $\mathbb{R}^4$. What is its dimension?
b) Find all values of a and b for which the implicit function theorem guarantees that $F^{-1}(a, b)$ is a smooth submanifold of $\mathbb{R}^4$.

(3) let $X = S^2/\sim$ where $S^2 = \{(x, y, z) \in \mathbb{R}^3 :\ x^2 + y^2 + z^2 = 1\}$, and the equivalence relation is $(1, 0, 0) \sim (-1, 0, 0)$; $(0, 1, 0) \sim (0, -1, 0)$. Find $\pi_1(X)$ and $H_2(X)$.

(4) Let

$$X_{(x^1,x^2,x^3,x^4)} = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2} + x^4\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^4}$$

be a vector field in $\mathbb{R}^4$
a) Show that X is tangent to the sphere $S^3 = \{x \in \mathbb{R}^4 :\ |x|^2 = 1\}$.
b) Let $\phi(t)$ be the one-parameter group of diffeomophisms of S^3 that is generated by X. Compute the diffeomorphism $\phi(\pi)$.

(5) Let ω be a *closed* two-form on the four-dimensional sphere S^4.
a) Prove that the form $\omega \wedge \omega$ is exact.
b) Compute

$$\int_{S^4} \omega \wedge \omega.$$

Justify your answer.

(6) Let

$$E = \{(z_1, z_2) \in \mathbb{C}^2 :\ z_1z_2 = 1.\}$$

Define a mapping

$$\pi :\ E \to \mathbb{C}^*\ (= \mathbb{C} \setminus \{0\})$$

by the formula

$$\pi(z_1, z_2) = \frac{z_1}{z_2}.$$

Prove that π is a covering and find its group of deck transformations