Geometry-Topology Qualifying Exam

January 2020

Problem 1
Use contour integration to evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{5 - 4\sin\theta}.$$

Problem 2
Let x^1, x^2, x^3 be Cartesian coordinates on a Euclidean three-dimensional space $M = \mathbb{R}^3$ and let $\mathcal{X}(M)$ denote the space of smooth vector field on it. For any vector field X define one- and two-forms ω^1_X and ω^2_X by

$$\omega^1_X(Y) = (X, Y), \qquad \omega^2_X(Y, Z) = (X, Y, Z).$$

Here (X, Y) is the dot product of X and Y, and (X, Y, Z) is the triple product of X, Y, and Z (which equals to the determinant of the matrix with columns X, Y, and Z).

Consider operations $curl : \mathcal{X}(M) \to \mathcal{X}(M)$, and $div : \mathcal{X}(M) \to C^\infty(M)$ defined by the formulas

$$d\omega^1_Y = \omega^2_{curl\, Y}, \qquad d\omega^2_Z = (div\, Z) dx^1 \wedge dx^2 \wedge dx^3.$$

a) Demonstrate that the exterior product of one-forms corresponds to the cross product of vector fields, by proving $\omega^1_X \wedge \omega^1_Y = \omega^2_{X \times Y}$.

b) Demonstrate that the exterior product of a one-form and a two-form corresponds to the dot product of vector fields: $\omega^1_X \wedge \omega^2_Y = (X, Y) dx^1 \wedge dx^2 \wedge dx^3$.

c) Express the following relation in terms of differential forms:

$$div(X \times Y) = (curl\, X, Y) - (curl\, Y, X).$$

d) Prove this relation using its differential form presentation.

Problem 3

Consider the following map

$$F : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x(x-y+z) - yz \\ x^2 + y^2 - 2xy \\ y - z \end{pmatrix}.$$

Let $C = \{(u, v, w) \in \mathbb{R}^3 | u^2 + v^2 = 1, w = 0\}$ denote a unit circle in $\mathbb{R}^3$.

Is its preimage $F^{-1}(C)$ of C a regular submanifold?

If it is, what is its dimension? If it is not, why not?

Problem 4

Compute the fundamental group of $\mathbb{R}^3$ with two lines removed. To receive full credit, you should consider different cases.

Problem 5

Let $\pi\colon S^2 \to \mathbb{RP}^2$ be the natural projection, sending a point $(x, y, z) \in S^2$ to the point $(x : y : z)$ in $\mathbb{RP}^2$. Show that the map π does not admit a continuous right inverse, i.e. there exists no continuous map $\phi\colon \mathbb{RP}^2 \to S^2$ such that $\pi \circ \phi = \mathrm{id}$.

Problem 6

Let M be an n-dimensional manifold, and let $\phi\colon M \to M$ be a continuous map such that the induced map $\phi_*\colon H_n(M, \mathbb{Z}) \to H_n(M, \mathbb{Z})$ is given by $\phi_*(\xi) = -\xi$. Define the space X as the quotient of $M \times [0, 1]$ by the following equivalence relation: $(x, 0) \sim (\phi(x), 1)$.

1. Let $\pi\colon M \times [0, 1] \to X$ be the quotient map. Define the set A as an open neighborhood of $X_1 = \pi(M \times [0, 1/2])$ which deformation retracts to X_1. Similarly, let B be an open neighborhood of $X_2 = \pi(M \times [1/2, 1])$ which deformation retracts to X_2. Describe the map

$$H_n(A \cap B, \mathbb{Z}) \to H_n(A, \mathbb{Z}) \oplus H_n(B, \mathbb{Z})$$

in the Mayer-Vietoris sequence for the decomposition $X = A \cup B$.

2. Hence show that $H_{n+1}(X, \mathbb{Z}) = 0$.