Geometry-Topology Qualifying Exam

January 2021

Problem 1

Find a conformal mapping, $w = f(z)$, which maps $\{z : 1 < |z| < 2\} \setminus \{z : -2 < Re(z) < -1, Im(z) = 0\}$ (an annulus minus a line segment) onto a rectangle.

Problem 2

(a) Show that $SL(2, \mathbb{R})$, the set of real 2×2 matrices with determinant one, is an embedded submanifold of the linear space of all real 2×2 matrices.

(b) Find the tangent space for $SL(2, \mathbb{R})$ at the identity matrix.

(c) Find a coordinate chart for $SL(2, \mathbb{R})$ in the vicinity of the identity matrix.

Problem 3

Let $U \subset \mathbb{R}\mathrm{P}^1$ be the subset $\{[z_1 : z_2] \in \mathbb{R}\mathrm{P}^1 \mid z_2 \neq 0\}$. The function $x = z_1/z_2$ defines a coordinate on U. Consider the one form $(1 + x^2)^k dx$ in U, where $k \in \mathbb{R}$. Find the values of k for which this one form extends to a smooth one form on the whole space $\mathbb{R}\mathrm{P}^1$.

Problem 4

(a) Compute the fundamental group of $\mathbb{R}^3 \setminus l$, where l is the z-axis.

(b) Compute the fundamental group of $\mathbb{R}^3 \setminus S^1$, where S^1 is the unit circle in the xy-plane.

Problem 5 Do one of the following two problems:

(a) Let $M \subset \mathbb{R}^3$ denote a compact three-dimensional submanifold with a smooth boundary. Let n denote the outward pointing unit normal vector along the boundary of M. Suppose that v is a smooth vector field defined on M. The divergence theorem from vector calculus states that

$$\int_{\partial M} (v \cdot n) dA = \int_M div(v) dV$$

Formulate this result in terms of differential forms, and explain why it is a special case of the general differential forms version of Stokes's theorem.

(b) Let M denote a compact manifold. DeRham's theorem asserts that there is a natural isomorphism

$$H^k_{DR}(M;\mathbb{R}) \to H_k(M;\mathbb{R})^*$$

for each k (where H_k denotes singular or simplicial homology, and V^* denotes the dual space of a vector space V). What is the map, and why is it well-defined?

Problem 6

Give examples of the following (with brief justification), or prove that none exists:

(a) a compact two-manifold with $H_2(X,\mathbb{Z}) = 0$.

(b) a two-form ω on $\mathbb{R}^3 \setminus \{0\}$ such that

$$\int_{S^2} \omega \neq 0$$

(c) a non-normal covering space of the figure eight.

(d) a space which has $F_2 \times (\mathbb{Z}/2\mathbb{Z})$ as fundamental group, where F_2 is the free group on two generators.