Geometry/Topology Qualifying Exam, January 2023

1) Let $f(x+iy) = u(x,y) + iv(x,y)$ be a function with real part u and imaginary part v.

a) If f is holomorphic, show u and v are harmonic, i.e.,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}.$$

You can use that the real and imaginary parts of any holomorphic function are infinitely differentiable, so there is no need to explain why the said second derivatives exist.

b) If $u(x,y) = 2x - y$ find $v(x,y)$ such that $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic.

2) Consider the map $F : \mathbb{R}^3 \to \mathbb{R}^2$ given by $F(x,y,z) = (x^2 - z^2, y^2 - z)$.

a) For what values of $(\alpha, \beta) \in \mathbb{R}^2$ does the Implicit Function Theorem guarantee that $F^{-1}(\alpha,\beta)$ is a one-dimensional regular submanifold of $\mathbb{R}^3$?

b) Describe the tangent space to the submanifold $F^{-1}(3,1)$ at the point $(2,1,0)$ as a subspace of $T_{(2,1,0)}\mathbb{R}^3$.

3) Consider the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\Phi(\theta,\phi) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi).$$

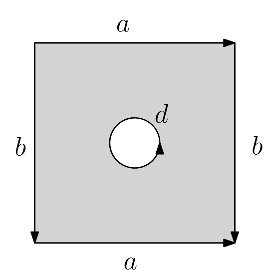
a) Show that $\frac{\partial}{\partial\theta}$ is Φ-related to a smooth vector field on $\mathbb{R}^3$, i.e., there exists a smooth vector field X on $\mathbb{R}^3$ such that

$$X_{\Phi(\theta,\phi)} = d\Phi\left(\left.\frac{\partial}{\partial\theta}\right|_{(\theta,\phi)}\right)$$

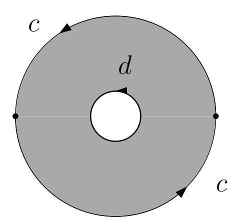
for all $(\theta,\phi) \in \mathbb{R}^2$.

b) Find all points $(\theta,\phi) \in \mathbb{R}^2$ at which the form $\Phi^*(dx \wedge dz) \in \Omega^2(\mathbb{R}^2)$ is equal to zero.

4) Recall that the connected sum of closed surfaces A and B is gotten by removing a 2-disk D^2 from each of A and B and identifying the boundary circles. Let X denote the connected sum of the torus $S^1 \times S^1$ and $\mathbb{RP}^2$. The connected sum is indicated below with identifications a, b, c, d specified.



$S^1 \times S^1 \setminus D^2$



$\mathbb{RP}^2 \setminus D^2$

a. Give a presentation of the fundamental group of X.

b. Consider the map $\phi : X \to \mathbb{RP}^2$ gotten by identifying all $S^1 \times S^1 \setminus D^2$ on the left side to a single point. Explain why ϕ is not homotopic to a constant map.

5) Let M be a connected, closed, orientable n-dimensional manifold. Recall that $H_k(M) = 0$ for $k > n$ and $H_n(M) \cong \mathbb{Z}$, where the group is generated by a fundamental class (a singular n-chain with simplex images that fill out M and whose orientations are consistent). Let $p \in M$. We will consider the decomposition of $M = U \cup V$ where $U = M \setminus \{p\}$ and V is an open set containing p and homeomorphic to an n-disk such that $U \cap V$ is homeomorphic to a cylinder $S^{n-1} \times I$, where $I \subset \mathbb{R}$ denotes an open interval.

a) In the Mayer-Vietoris sequence for this decomposition, if we suppose that $H_n(M) \to H_{n-1}(U \cap V)$ is an isomorphism, where $n = \dim M$, compute the homology groups of $M \setminus \{p\}$ in terms of the homology groups of M.

b) Show that $H_n(M) \to H_{n-1}(U \cap V)$, where $n = \dim M$, is an isomorphism using the definition of the connecting homomorphism.

6) Suppose ω is a smooth, closed 2-form on S^4. Show that $\omega \wedge \omega$ vanishes somewhere.