GEOMETRY/TOPOLOGY QUALIFYING EXAM

JANUARY 2024

Please show all of your work. GOOD LUCK!

1) Evaluate the following integral

$$\int_0^\infty \frac{\ln x}{4+x^2}dx.$$

Justify all steps.

2) Let the mapping $F : \mathbb{R}^4 \to \mathbb{R}^2$ be given by the formulas

$$F(w,x,y,z) = (w^2+x^2+y^2+z^2, w^2+x^2-2y^2-3z^2).$$

Find all values of $a > 0$ such that $F^{-1}(a,1)$ is a smooth two-dimensional submanifold of $\mathbb{R}^4$.

3) Evaluate

$$\int_S (x^2+y^2+z^2)^{-3/2}(xdy\wedge dz - ydx\wedge dz + zdx\wedge dy),$$

where S is the surface in $\mathbb{R}^3$ given by the equation

$$x^2+y^4+z^6 = 5.$$

4) The boundary of a Möbius strip M is a circle: $\partial M = S^1$. A Klein bottle K can be viewed as a union of two Möbius strips, M_1 and M_2, with their boundary circles identified (by the identity map). Introduce open subspaces $U_1 \supset M_1$ and $U_2 \supset M_2$ of K and use this decomposition of the Klein bottle to compute its integer homology via the Mayer-Vietoris sequence.

   Be sure to compute the homology in all degrees.

5) Compute the fundamental group of a two-sphere with three of its points identified.

6) Recall, that for $A \subset X$ with the inclusion map $i : A \to X$, a map $r : X \to A$ is a *retraction* if $r \circ i = id_A$.

   Let X be a two-dimensional disk with two of its distinct boundary points identified, and let $A = \partial X$ be its boundary (a bouquet of two circles). Is there a retraction of X to ∂X?