ALGEBRA QUALIFYING EXAMINATION

AUGUST 2024

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A. Put

$$A = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 0 & & & \\ & & 0 & 1 & & \\ & & & 0 & 0 & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 1 & & \\ & 0 & 0 & 0 & 1 & \\ & & 0 & 0 & 0 & 1 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}.$$

Are A and B similar? Explain why.

1B. Suppose V is a finite-dimensional vector space over $\mathbb{R}$, and let $\phi\colon V \to V$ be a linear operator such that $\phi^3 = \phi$. Let $V_\lambda = \{v \in V \mid \phi(v) = \lambda v\}$ be the λ-eigenspace of ϕ. Prove that $V = V_0 \oplus V_1 \oplus V_{-1}$.

2A. Let A be an abelian group. Denote by $2A$ the subgroup generated by elements of the form $2a$, $a \in A$.
(1) Prove that if A is finitely generated, then $A/2A$ is finite.
(2) Give an example to show that if $A/2A$ is finite, A itself might not be finitely generated. You need to justify your claims.

2B. Suppose G is a simple group of order 2025. Prove that G has a subgroup of order 5 which is contained in at least two subgroups of order 25.

3A. Consider the ring $\mathbb{C}[x, y]$.
(1) Show that $\mathfrak{m} = (x, y)$ is a maximal ideal and it is not principal.
(2) What is the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over $\mathbb{C}$? You need to justify your claims. (Note: the notation $\mathfrak{m}^2$ means the ideal generated by elements of the form ab where $a, b \in \mathfrak{m}$.)

3B. Prove that $\mathbb{Z}[\sqrt{-7}]$ is not a PID.

4A. (1) Show that the extension $\mathbb{Q}(\sqrt[4]{5})/\mathbb{Q}$ is not Galois.
(2) Compute the Galois group of $x^4 - 5$ over $\mathbb{Q}(\sqrt{-1})$.
(3) Show that the Galois group of $x^4 - 5$ over $\mathbb{Q}$ is not abelian. (Hint: find two non-commuting elements; you do not need to actually compute this Galois group.)

4B. Prove that $\mathbb{Q}(\sqrt{2 + \sqrt{2}})/\mathbb{Q}$ is a Galois extension with cyclic Galois group. (Hint: Show that the minimal polynomial of $\sqrt{2 + \sqrt{2}}$ over $\mathbb{Q}$ splits over $\mathbb{Q}(\sqrt{2 + \sqrt{2}})$.)

5A. Let R be a commutative unital ring. Let M, N and P be R-modules, and $f : M \to N$ be a homomorphism.
(1) Show that there is a unique homomorphism $f \otimes 1_P : M \otimes_R P \to N \otimes_R P$ which sends $m \otimes p$ to $f(m) \otimes p$, for all $m \in M$ and $p \in P$.
(2) Show that if f is surjective, then so is $f \otimes 1_P$.
(3) Give an example of R, M, N, P and f as above, such that f is injective but $f \otimes 1_P$ is not. You do not need to give a detailed justification, but you need to clearly state what your R, M, N, P and f are and what $M \otimes_R P, N \otimes_R P$ and $f \otimes 1_P$ are.

5B. Endow the Abelian group $\mathbb{R} \oplus \mathbb{R}$ with a ring structure by setting

$$(a, a')(b, b') = (ab, ab' + a'b).$$

(1) Prove that the map $a_0 + a_1 x + \cdots \mapsto (a_0, a_1)$ is a surjective ring homomorphism $\mathbb{R}[x] \to \mathbb{R} \oplus \mathbb{R}$.
(2) Using the result of the previous part or otherwise, prove that any finitely generated module over $\mathbb{R} \oplus \mathbb{R}$ is a direct sum of cyclic modules.