Analysis Qualifying Exam — August 2024

The University of Arizona

Directions: Read carefully and provide complete, well written, well explained and organized answers. State any basic results or major theorems used, verifyig their applicability.

1. Let (M, d) be a metric space. A function $f : M \to \mathbb{R}$ is called Lipschitz if there exists a constant $C > 0$ such that
$$|f(x) - f(y)| \leq Cd(x, y)$$
for every $x, y \in M$.

   Give a counter example for the following statement:
   If $\{f_n\}_{n\geq 0}$ is a sequence of Lipschitz functions on M that converges uniformly to a function f, then f is Lipschitz.

2. Find the following limit
$$\lim_{n\to\infty} \sqrt{n} \int_0^1 \left(1 - x^2\right)^n dx.$$
Justify all steps.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space. If f is a non-negative, μ-integrable function, then
$$\nu(A) := \int_A f\, d\mu$$
is a measure over $\mathcal{A}$. Prove that if g is a non-negative function that is integrable with respect to ν, then fg is integrable with respect to μ and
$$\int g\, d\nu = \int fg\, d\mu.$$

4. Suppose that f is monotone, absolutely continuous on $[0, 1]$; for $A \subset [0, 1]$ we let $f(A) = \{f(x) : x \in A\}$. Prove that if A has Lebesgue measure 0, then $f(A)$ has Lebesgue measure 0.

5. Suppose that $p > 1$ and q are Hölder conjugates. Let f be an absolutely continuous function on $[0, 1]$ with $f' \in L^p$, and $f(0) = 0$. Prove that if $g \in L^q$ then
$$\int_0^1 |fg| dx \leq (1/p)^{1/p} \|f'\|_p \|g\|_q.$$

6. Suppose that H is a separable Hilbert space, $\{e_n\}_{n\geq 0}$ is an orthonormal basis for H, and $\{f_n\}_{n\geq 0}$ is an orthonormal set such that $\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1$. Prove that $\{f_n\}_{n\geq 0}$ is a basis.

   Hint. Suppose that there exists a unit vector that is orthogonal to all vectors f_n. Show that this leads to a contradiction.