Geometry-Topology Qualifying Exam

Fall 2024

Problem 1

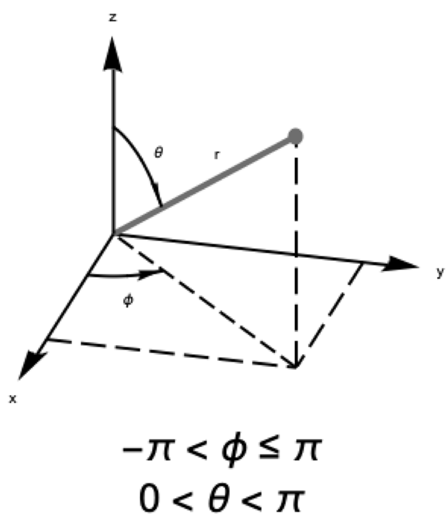
Evaluate the following integral

$$\oint_C \frac{z^2}{e^{\pi z}+1} dz,$$

where $C := \{z \in \mathbb{C} : |z| = 2\}$ is a radius two circle centered at the origin with the counterclockwise orientation.

Problem 2

Find a curve on the surface of the unit two-sphere which forms a 45 degree angle with every meridian it intersects. In other words, what path do you trace if you start on the Equator and keep constantly moving in the North-East direction?



Note: A *meridian* is any great semicircle connecting the two poles, i.e. a curve with a fixed value of ϕ and θ varying from 0 to π.

Problem 3

Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F : M \to \mathbb{R}^k$ for any positive integer k.

Problem 4

Suppose the smooth maps $f : X \to Y$ and $g : X \to Y$ are homotopic and X is a smooth compact manifold of dimension n. (Note: Y is smooth, but not necessarily compact.)

Prove that for all closed n-form ω on Y

$$\int_X f^*\omega = \int_X g^*\omega.$$

Problem 5

Compute the fundamental group of $T^2 \# \mathbb{R}P^2$, the connected sum of a two-torus T^2 and a projective plane $\mathbb{R}P^2$ and prove your result. Recall, that a *connected sum* of two manifolds results from deleting a disk in each of them and identifying the hems.

Problem 6

Let Γ be a closed simple curve in a three-sphere S^3. Find the reduced homology groups of its complement $X = S^3 \setminus \Gamma$ and prove that this result does not depend on the knot formed by Γ.

(Hint: You might want to use the Mayer-Vietoris theorem.)