ALGEBRA QUALIFYING EXAMINATION

JAN 2025

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A Let A, B be $n \times n$ matrices. (1) If A is invertible, show that AB is similar to BA. (2) Give a counterexample when A is not invertible.

1B Suppose J is a real 2×2 matrix such that $J^2 = -I$, where I is the identity matrix. Prove that J is similar to the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

2A Show that the additive group $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the multiplicative group of all roots of unity.

2B Prove that a group of order 255 must have a subgroup of order 15.

3A Let p be a prime. Define a subring R of $\mathbb{Q}$ by

$$\left\{ \frac{a}{b} \mid a, b \in \mathbb{Z},\ p \nmid b \right\}.$$

(1) Show that R is a PID. (2) Find all prime ideals of R.

3B Consider the ideal $\langle 1+\sqrt{-3}, 1-\sqrt{-3} \rangle$ in $\mathbb{Z}[\sqrt{3}]$. Determine whether it is (1) principal (2) maximal.

4A (1) Let p be a prime. Show that the Galois group of $\mathbb{Q}(e^{2\pi\sqrt{-1}/p})/\mathbb{Q}$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{\times}$. (2) Use the first part and Galois theory to find a degree 7 Galois extension of $\mathbb{Q}$.

4B Suppose that $f(x) \in \mathbb{Q}[x]$ is an irreducible polynomial of degree 7 whose Galois group over $\mathbb{Q}$ is Abelian. Prove that f has seven distinct real roots.

5A Let A be a ring with multiplicative identity 1, and A^{opp} the opposite ring, i.e. the multiplication $\cdot_{\text{opp}}$ in A^{opp} is given by $a \cdot_{\text{opp}} b = ba$ where the right hand side is the multiplication in A. View A as a left module over A itself. Show that there is a ring isomorphism

$$\text{Hom}_A(A, A) \simeq A^{\text{opp}}.$$

5B Suppose R is a ring which is finitely generated as an Abelian group. Prove that for any $\alpha \in R$ there exists a non-zero polynomial $f \in \mathbb{Z}[x]$ such that $f(\alpha) = 0$.