Analysis qualifier — January 2025

The University of Arizona

Directions: Read carefully and provide complete, well written, well explained and organized answers. <u>**State any basic results or major theorems used, verifying their applicability.**</u>

1. Suppose f and g are continuous on $[a,b]$ and such that $\int_a^b f = \int_a^b g$. Prove that there exists $x \in (a,b)$ such that $f(x) = g(x)$.

2. Let $g : \mathbb{R} \to \mathbb{R}$ be integrable and $f : \mathbb{R} \to \mathbb{R}$ be bounded, measurable, and continuous at 1. Prove that
$$\lim_{n\to\infty} \int_{-n}^{n} f(1 + x/n^2) g(x)\, dx$$
exists and compute its value (the answer must be written in terms of the functions f and g).

3. Suppose $1 \le r < p < s < \infty$ and (X, A, μ) is a measure space. Prove that $L^r(X) \cap L^s(X) \subset L^p(X)$.

4. An operator A takes continuous functions on the interval $[0,1]$ to functions on $[0,1]$ by the formula
$$Au(x) = \int_0^1 \frac{u(y)}{|x-y|^{1/3}} dy.$$
Prove that it can be extended to a bounded operator from $L^4([0,1])$ to $L^1([0,1])$.

5. Suppose H is a Hilbert space with norm $\|\cdot\|$. Suppose that $\|x_n\| \to \|x\|$ and $Tx_n \to Tx$ for every bounded linear functional T. Prove that $\|x_n - x\| \to 0$.

6. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, f and g be real-valued measurable functions on X, and $f \in L^1(X)$. Suppose that
$$\mu(\{x : f(x) > t\}) = \mu(\{x : g(x) > t\}) \qquad \text{for every } t \in \mathbb{R}.$$
Prove that $g \in L^1(X)$ and
$$\int_X f(x) d\mu = \int_X g(x) d\mu.$$
[Hint: try expressing $f(x) = \int_0^{f(x)} dt$]