Geometry-Topology Qualifying Exam

Spring 2025

Problem 1

Find the Fourier transform $F : \mathbb{R} \to \mathbb{R}$ of the real-valued function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \frac{1}{1+x^2}.$$

I.e. compute

$$F(\omega) := \int_{-\infty}^{+\infty} \frac{e^{-i\omega x}}{1+x^2} dx.$$

Problem 2

Let M and N be smooth manifolds of the same dimension, and assume M is compact. Let $f : M \to N$ be a smooth map and let $y \in N$ be a regular value of f. Prove that the level set $f^{-1}(\{y\})$ is a finite set.

Problem 3

Consider the algebra $\mathcal{G}$ generated by the following two vector fields on a plane

$$U = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad V = (1 + x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y},$$

with the product given by the commutator of vector fields.

Prove that

a) $\mathcal{G}$ is a three-dimensional algebra and

b) that it is isomorphic to the algebra $\mathcal{H} = (\mathbb{R}^3, \times)$ of three-vectors with a vector (cross) product.

Problem 4

Let $\mathring{D}^2_R = D^2_R \setminus \{0\}$ denote a punctured disk of radius R and let

$$\omega = \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy$$

be a one-form on $\mathring{D}^2_R$.

a) Compute $d\omega$.

b) Compute $\int_{S^1_R} \omega$, the integral of ω over a circle of radius $R > 0$ centered at the origin (with counterclockwise orientation).

c) Does $\int_{\mathring{D}^2_R} d\omega = \int_{S^1_R} \omega$? Why?

Problem 5

Consider the solid torus T in $\mathbf{R}^3$, obtained by rotating the disc

$$D = \{(x, y, z) \mid (x-2)^2 + z^2 \leq 1, y = 0\}$$

around the z-axis. Let X be the topological space obtained by removing from T two circles lying in the xy plane: $x^2 + y^2 = \left(\frac{3}{2}\right)^2$ and $x^2 + y^2 = \left(\frac{5}{2}\right)^2$.

Find the fundamental group of X.

Problem 6

Let d be a natural number greater than one. Prove that for a connected sum $X\#Y$ of d-dimensional manifolds X and Y,

$$H_1(X\#Y) \simeq H_1(X) \oplus H_1(Y).$$

(The connected sum of two spaces is obtained by cutting out a small ball out of each space and identifying the resulting boundary spheres.)