Kähler Manifolds and their Deformations

Description

This course will give an introduction to the geometry of complex and Kähler manifolds, followed by an introduction to the deformation theory of compact complex manifolds.

The first part of the course will introduce the basic objects and tools of complex differential geometry, including complex manifolds, holomorphic vector bundles, Dolbeault cohomology, Kähler metrics, and Hodge theory on compact Kähler manifolds.

The second part of the course will introduce the classical deformation theory of Kodaira and Spencer. Topics will include families of complex manifolds, infinitesimal deformations and the Kodaira–Spencer map, and existence and completeness of deformation families. Examples such as Riemann surfaces, complex tori, and complex hypersurfaces will be discussed throughout.

Texts

- John M. Lee, *Introduction to Complex Manifolds*, Graduate Studies in Mathematics 244, American Mathematical Society, 2024.
- Kunihiko Kodaira, *Complex Manifolds and Deformation of Complex Structures*, Classics in Mathematics, Springer.

Prerequisites

Students should have completed MATH 534A–B or have equivalent background. MATH 520B or equivalent background is recommended. Some familiarity with functional analysis is also expected.

Learning Outcomes

By the end of the course, students should be able to:

1. Work with the basic geometric structures associated with a complex manifold, including holomorphic tangent and cotangent bundles, (p, q)-forms, and the operators ∂ and $\bar{\partial}$.
2. Explain the defining properties of Kähler manifolds and derive fundamental consequences of the Kähler condition, including the Hodge decomposition and the $\partial\bar{\partial}$-lemma.
3. Describe smooth and holomorphic families of complex manifolds, and understand the construction of the associated Kodaira–Spencer map.
4. Explain the basic ideas underlying the Kodaira–Spencer existence and completeness theorems and the local structure of deformation spaces.

5. Compute or describe deformation spaces in basic examples.

6. Explain the statement and main ideas behind the stability of the Kähler condition under small deformations.

Approximate Schedule

Weeks 1–2. **Introduction to complex manifolds.** Introduction to complex manifolds, holomorphic maps, and holomorphic vector bundles.

Weeks 3–5. **Differential forms and cohomology.** Decomposition of forms according to bidegree, Dolbeault cohomology, sheaves, Čech cohomology, and the Dolbeault isomorphism.

Weeks 6–7. **Kähler geometry and Hodge theory.** Definitions and examples of Kähler metrics, the Kähler identities, elliptic operators, and the Hodge decomposition.

Spring Break.

Week 8. **Families of complex manifolds.** Families of complex manifolds, Ehresmann's theorem, and examples of deformations.

Weeks 9–10.
Infinitesimal deformations and integrability. The Kodaira–Spencer map, relation between the Čech cocycle description and the Dolbeault formulation of infinitesimal deformations, and the integrability condition in both descriptions.

Weeks 11-12.
Existence of deformations. Outline of the proof of the Kodaira–Spencer existence theorem in the unobstructed case, including the analytic ideas entering the construction.

Weeks 13–14.
Completeness and local deformation spaces. Complete and versal families, the Kodaira–Spencer completeness theorem, and the local structure of deformation spaces.

Week 15. **Stability under deformation.** Outline of the proof of the Kodaira–Spencer theorem that sufficiently small deformations of a compact Kähler manifold remain Kähler.