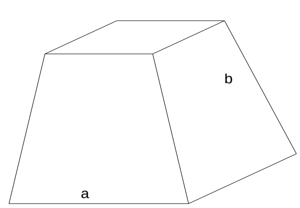
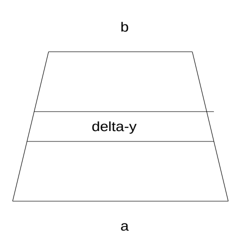
The Pyramid Problem
Math 129

In 1878, a pair of brothers, the soon-to-become-infamous thieves Ahmed and Mohammed Abd er-Rassul, stumbled upon the ancient Egyptian burial site in the Valley of the Kings, at Dier el-Bahri. They quickly built a thriving business selling stolen relics, one of which was a mathematical papyrus; one of the brothers sold it to the Russian Egyptologist V. S. Golenishchev in 1893. He in turn gave it to the Moscow Museum of Fine Arts in 1912. There it languished until a complete translation was made in 1930, at which time the scholarly world learned how mathematically advanced the ancient Egyptians really had been.
In particular, the fourteenth problem of the Moscow Mathematical Papyrus, as it is now called, is a specific numerical example of how to calculate the volume, V of a truncated square pyramid, a geometrical object known as a frustrum of a pryamid. The example strongly indicates that the ancient Egyptians knew the formula:

$$V = \frac{1}{3}h(a^2 + ab + b^2)$$

where a and b are the edge lengths of the top and bottom squares respectively, and h is the height. One historian of science has called this knowledge "breath-taking," and the "masterpiece of Egyptian geometry." The derivation of this formula is a routine exercise for anyone who has had freshman calculus, but it is not clear how the Egytians got it without calculus.[1]
Let's do this routine exercise.





[1]from An Imaginary Tale, The Story of $\sqrt{-1}$, by Paul J. Nahine

From the sketch, it is clear that subdivision in the y direction is the way to go. We need the cross section at any point in terms of y, which is related to x by a straight line running through the points $(a/2, 0)$ and $(b/2, h)$. Thus:

$$\begin{aligned} \frac{y-0}{x-a/2} &= \frac{h-0}{b/2-a/2} && (1) \\ \text{so } \frac{1}{2}(b-a)y &= hx - \frac{h}{2}a && (2) \\ \frac{1}{2}(b-a)y + \frac{h}{2}a &= hx && (3) \\ \text{and } x &= \frac{1}{2h}(b-a)y + \frac{a}{2} && (4) \end{aligned}$$

The side of the square cross-section of the frustum of the pryamid is twice this x giving us an approximating sum

$$\sum_{i=1}^{n}(2x_i)^2\Delta y_i = \sum_{i=1}^{n} 4x_i^2\Delta y_i$$

which leads, using (4) above to

$$\begin{aligned} \int_0^h 4x^2 dy &= 4\int_0^h (\frac{b-a}{2h})^2 y^2 + 2(\frac{a}{2})(\frac{b-a}{2h})y + \frac{a^2}{4}dy \\ &= 4\left((\frac{(b-a)^2}{4h^2})\frac{y^3}{3} + a(\frac{(b-a)}{2h}\frac{y^2}{2}) + \frac{a^2}{4}y\Big|_0^h \right) \\ &= \frac{(b-a)^2}{h^2}\frac{h^3}{3} + (ab-a^2)h + a^2h \\ &= \frac{b^2h}{3} - \frac{2abh}{3} + \frac{a^2h}{3} + \frac{3abh}{3} - \frac{3a^2h}{3} + \frac{3a^2h}{3} \\ &= \frac{b^2h}{3} + \frac{abh}{3} + \frac{a^2h}{3} = V = \frac{1}{3}h(a^2+ab+b^2) \end{aligned}$$

Routine, as the ancient Egyptians knew!