Derivation of Simpson's Rule
Math 129

1. The form of Simpson's Rule given in the book is

$$\text{SIMP}(n) = \frac{1}{3}\left[2\text{MID}(n) + \text{TRAP}(n)\right]$$

But

$$\text{TRAP}(n) = [\text{LEFT}(n) + \text{RIGHT}(n)]/2$$

So SIMP(n) is really

$$\frac{1}{3}\left[(\text{LEFT}(n) + \text{RIGHT}(n))/2 + 2\text{MID}(n)\right]$$

or

$$\frac{1}{3}(\text{LEFT}(n)/2 + \text{RIGHT}(n)/2 + 2\text{MID}(n))$$

2. So this suggests a refinement of the Riemann sum that approximates

$$\int_a^b f(x)dx.$$

We will want to use three points in each subinterval of the partition for the Riemann Sum, the left and right-hand endpoints, and the midpoint. To use all three points we will fit a parabola through them. So the first step is to find the area under a piece of a parabola in terms of these three points. We shall assault this problem in steps.

3. The book claims that if $f(x)$ is quadratic, then this formula applies:

$$\int_a^b f(x)dx = \frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right)$$

where $h = b - a$ and the midpoint is $m = (b - a)/2$. Let us verify this, and then apply it to the parabola approximates the function over one strip in the Riemann Sum.

4. A general parabola looks like $Ax^2 + Bx + C1$ and we need to get

$$\int_a^b Ax^2 + Bx + C1dx = A\int_a^b x^2dx + B\int_a^b xdx + C\int_a^b 1dx$$

a. If $f(x) = 1$ then

$$\int_a^b f(x)dx = (b-a)$$

and

$$\frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right) = \frac{(b-a)}{3}\left(\frac{1}{2} + 2 + \frac{1}{2}\right)$$

which is $(b-a)$, so the formula works for $f(x) = 1$.

b. We do the same for $f(x) = x$.

$$\int_a^b x dx = \frac{b^2 - a^2}{2}$$

and

$$\begin{aligned}
\frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right) &= \frac{b-a}{3}\left(\frac{a}{2} + 2\frac{a+b}{2} + \frac{b}{2}\right) \\
&= \frac{b-a}{3}\left(\frac{a}{2} + a + b + \frac{b}{2}\right) \\
&= \frac{b-a}{3}\left(\frac{3}{2}a + \frac{3}{2}b\right) \\
&= \frac{(b-a)(b+a)}{2} \\
&= \frac{b^2 - a^2}{2}
\end{aligned}$$

and so the formula works for $f(x) = x$.

c. For $f(x) = x^2$,

$$\int_a^b f(x)dx = \frac{b^3 - a^3}{3}$$

and

$$\frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right) = \frac{b-a}{3}\left(\frac{a^2}{2} + 2(\frac{a+b}{2})^2 + \frac{b^2}{2}\right)$$

$$\begin{aligned}
&= \frac{b-a}{3}\left(\frac{a^2}{2} + \frac{a^2+2ab+b^2}{2} + \frac{b^2}{2}\right) \\
&= \frac{b-a}{3}\left(\frac{2a^2+2ab+2b^2}{2}\right) \\
&= \frac{b-a}{3}(a^2+ab+b^2) \\
&= \frac{b^3-a^3}{3}
\end{aligned}$$

and again it works.
5. For our general quadratic, $f(x) = Ax^2 + Bx + C$, we have

$$\int_a^b Ax^2 + Bx + C1dx = A\int_a^b x^2dx + B\int_a^b xdx + C\int_a^b 1dx$$

and using the above results we get

$$\begin{aligned}
\int_a^b f(x)dx &= A\frac{h}{3}\left(\frac{a^2}{2} + 2m^2 + \frac{b^2}{2}\right) + B\frac{h}{3}\left(\frac{a}{2} + 2m + \frac{b}{2}\right) + C\frac{h}{3}(3) \\
&= \frac{h}{3}\left(\frac{Aa^2+Ba+C}{2} + 2(Am^2+Bm+C) + \frac{Ab^2+Bb+C}{2}\right) \\
&= \frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right)
\end{aligned}$$

and we get the same formula for the general quadratic that we got for the pieces.

6. Now we are home free. To get the Riemann sum for

$$\int_a^b f(x)dx$$

using the quadratic approximation, assuming $f(x)$ is *any* integrable function, we partition the interval $[a, b]$, and let $q_i(x)$ be the quadratic approximation to $f(x)$ on the subinterval $[x_i, x_{i+1}]$. Let m_i be the midpoint of this subinterval, and $\Delta x = h = (x_{i+1} - x_i)$. Our q_i passes through the endpoints and the midpoint of the subinterval. Then

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx \int_{x_i}^{x_{i+1}} q_i(x)dx$$

and using the results above on this second approximating integral we get

$$\frac{\Delta x}{3}\left(\frac{q_i(x_i)}{2} + 2q_i(m_i) + \frac{q_i(x_{i+1})}{2}\right)$$

Now we sum over all the subintervals

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} q_i(x)dx = \sum_{i=0}^{n-1}\frac{\Delta x}{3}\left(\frac{q_i(x_i)}{2} + 2q_i(m_i) + \frac{q_i(x_{i+1})}{2}\right)$$

Splitting the sum into two parts we get,

$$\begin{aligned}
&\frac{2}{3}\sum_{i=0}^{n-1} q_i(m_i)\Delta x + \frac{1}{3}\sum_{i=0}^{n-1}\left(\frac{q_i(x_i)}{2} + \frac{q_i(x_{i+1}}{2}\right)\Delta x \\
=\ &\frac{2}{3}\text{MID}(n) + \frac{1}{3}\text{TRAP}(n) \\
=\ &\frac{1}{3}\left[2\text{MID}(n) + \text{TRAP}(n)\right] \\
=\ &\text{SIMP}(n)
\end{aligned}$$

as was to be shown.

7. Who's afraid of a little algebra??