1. Sheaves and Ringed Spaces

1.1. **More examples of presheaves.**

(1) Let k be an algebraically closed field and X an algebraic set. Then
$$\mathcal{O}_X : U \mapsto \mathcal{O}_X(U) = \{\text{regular functions on } U\}$$
is a presheaf of k-algebras.

(2) Let X be a top. space and G an abelian group. The *constant presheaf* is
$$U \mapsto \begin{cases} G & U \neq \emptyset \\ 0 & U = \emptyset \end{cases}$$
together with the obvious restriction maps.

(3) The constant sheaf for G on X, $\underline{G}$ is
$$U \mapsto \{\text{continuous } G\text{-valued functions on } U\},$$
where G is given the discreet topology. Observe that if X is any locally connected space (like a manifold) then
$$\underline{G}(U) = \prod_{\text{conn. cpts. of } U} G.$$

(4) If $X = \mathbf{Z}_p$ or $X = \operatorname{Spec}(\prod_{\mathbf{Z}} k)$ with k a field, then $\underline{G}$ is hard to make as explicit as above.

(5) X any $\mathbf{C}$-manifold and $\mathcal{O}_X$ the presheaf of holomorphic functions on X. Then let
$$\mathcal{O}_X^\times : U \mapsto \{f \in \mathcal{O}_X(U) : f \neq 0 \text{ on } U\}.$$

1.2. **Examples of morphisms of (pre)sheaves.**

If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves on X, we define the presheaf $\ker\phi$ on X by
$$\ker\phi(U) = \ker\left(\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)\right).$$

(1) Let X be a $\mathbf{C}$-manifold and let $\mathbf{Z}(1)$ be the abelian group $2\pi\imath\mathbf{Z}$. Consider the exponential morphism $\exp : \mathcal{O}_X \to \mathcal{O}_X^\times$ given on the sections over U as $f \mapsto e^f$. Then one has $\ker\exp = \underline{\mathbf{Z}(1)}$, the constant sheaf on X associated to $\mathbf{Z}(1)$.

1.3. **Sheaves.**

Definition 1.4. Suppose $\mathcal{C}$ is a subcategory of the category of sets. Then a $\mathcal{C}$-valued presheaf $\mathcal{F}$ on X is a *sheaf* if for any open cover $\{U_\alpha\}$ of U and any collection of $s_\alpha \in \mathcal{F}(U_\alpha)$ that satisfy $s_\alpha\big|_{U_\alpha \cap U_\beta} = s_\beta\big|_{U_\alpha \cap U_\beta}$ for all α, β (as elements of $\mathcal{F}(U_\alpha \cap U_\beta)$), there is a *unique* $s \in \mathcal{F}$ with $s\big|_{U_\alpha} = s_\alpha$ for all α.

Roughly speaking, a sheaf is a presheaf with the additional property that local, compatible data uniquely determines global data. Observe that the compatibility criterion is vacuously satisfied if $U_\alpha \cap U_\beta = \emptyset$. We make the set of sheaves on X into a category by stipulating that morphisms of sheaves are just morphisms of presheaves as already defined.

Definition 1.5. If $\mathcal{F}$ is a sheaf, then a *subsheaf* of $\mathcal{F}$ is a subpresheaf $\mathcal{G} \subseteq \mathcal{F}$ that is a sheaf. By the uniqueness of glueing, glueing in $\mathcal{G}$ is the same as glueing in $\mathcal{F}$.

(1) The constant presheaf is *not* a subsheaf of the constant sheaf.

(2) Let X be a $\mathbf{C}^\infty$ manifold and Ω_X^k the presheaf of $\mathbf{C}^\infty$ k-forms (a sheaf). Let $\wedge_{\mathcal{O}_X}^k$ be the presheaf $U \mapsto \wedge_{\mathcal{O}_X(U)}^k(\Omega_X^1(U))$ (that is, the k th exterior power of the $\mathbf{C}$-vector space $\Omega_X^1(U)$). Then $\wedge_{\mathcal{O}_X}^k$ is *not* in general a sheaf. However, we have a map $\wedge_{\mathcal{O}_X}^k \to \Omega_X^k$ given on sections over U as $\omega_1 \wedge \ldots \wedge \omega_n \mapsto \omega_1 \wedge \ldots \wedge \omega_n$. This map is usually neither injective nor surjective.

Definition 1.6. Let X be a top. space, $U \subseteq X$ an open set, and suppose that $\mathcal{F}$ is a presheaf on X. Then we define the presheaf $\mathcal{F}\big|_U$ on U via $U \supseteq V \mapsto \mathcal{F}(V)$. Since U is open in X and V is open in U, we see that V is open in X so that this definition makes sense.

It is not hard to see that $\mathcal{F}\big|_U$ is a sheaf if $\mathcal{F}$ is.

(1) Let X be a $\mathbf{C}^\infty$ manifold and $\mathcal{F} = \mathcal{O}_X$. Then $\mathcal{F}\big|_U = \mathcal{O}_U$.
(2) Let X be a top. space and G an abelian group. Let $\mathcal{F} = (\underline{G}, X)$ be the sheaf $\underline{G}$ on X. Then $\mathcal{F}\big|_U = (\underline{G}, U)$ is the sheaf $\underline{G}$ of locally constant G-valued functions on U.
(3) Let $f : X' \to X$ be a continuous map of top. spaces and $\Gamma_{X'/X}$ the sheaf of sections
$$U \mapsto \{s : U \to f^{-1}(U) : f \circ s = \mathrm{id}_U\}.$$
If $W \subseteq X$ is open then $\Gamma_{X'/X}\big|_W = \Gamma_{f^{-1}(W)/W}$.

Definition 1.7. Let $f : X \to Y$ be a continuous map of topological spaces and $\mathcal{F}$ a presheaf on X. The *pushforward*, $f_*\mathcal{F}$ is the presheaf on Y defined by
$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)),$$
with the restriction maps $\rho_{f^{-1}(U),f^{-1}(V)} : (f_*\mathcal{F})(U) \to (f_*\mathcal{F})(V)$ inherited from $\mathcal{F}$.

Observe that pushforward if functorial. Indeed, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves on X and $f : X \to Y$ as above, we obtain a morphism $f_*\varphi : f_*\mathcal{F} \to f_*\mathcal{G}$ of presheaves on Y, where $(f_*\varphi)(U) : (f_*\mathcal{F})(U) \to (f_*\mathcal{G})(U)$ is just the map $\varphi(f^{-1}(U)) : \mathcal{F}(f^{-1}(U)) \to \mathcal{G}(f^{-1}(U))$.

Observe that if $\mathcal{F}$ is a sheaf on X and $f : X \to Y$ a continuous map of top. spaces then $f_*\mathcal{F}$ is a sheaf on Y. This is more or less a tautology. In complete detail, let $U \subset Y$ be open and $\{U_\alpha\}$ an open cover of U. Let $s_\alpha \in (f_*\mathcal{F})(U_\alpha)$ be a collection of sections with $s_\alpha\big|_{U_\alpha \cap U_\beta} = s_\beta\big|_{U_\alpha \cap U_\beta}$ as elements of $(f_*\mathcal{F})(U_\alpha \cap U_\beta)$. By the definition of f_*, we have $s_\alpha \in \mathcal{F}(f^{-1}(U_\alpha))$ with $s_\alpha\big|_{f^{-1}(U_\alpha \cap U_\beta)} = s_\beta\big|_{f^{-1}(U_\alpha \cap U_\beta)}$ as elements of $\mathcal{F}(f^{-1}(U_\alpha \cap U_\beta))$. Since $f^{-1}(U_\alpha \cap U_\beta) = f^{-1}(U_\alpha) \cap f^{-1}(U_\beta)$, we can rename (for psychological purposes really) $f^{-1}(U_\alpha) = V_\alpha \subseteq X$ and $f^{-1}(U) = V \subset X$ (so that V_α is a covering of V) and we see that the sheaf property of unique glueing for $\mathcal{F}$ gives an element $s \in \mathcal{F}(V) = (f_*\mathcal{F})(U)$ with $s\big|_{U_\alpha} = U_\alpha$.

(1) Let $\varphi : X \to Y$ be a $\mathbf{C}^\infty$ map of $\mathbf{C}^\infty$ manifolds, and let $\mathcal{O}_X, \mathcal{O}_Y$ be the sheaves of $\mathbf{C}^\infty$ functions on X and Y respectively. Now $f_*\mathcal{O}_X$ is a sheaf on $\mathcal{O}_Y$ and we have a map
$$\varphi^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$$
given by $g \mapsto g \circ \varphi$ as a map $\mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$ over any $U \subseteq Y$. Observe that this map makes sence as φ is $\mathbf{C}^\infty$ co that the composition $g \circ \varphi$ is $\mathbf{C}^\infty$ (and hence an element of $\mathcal{O}_X(\varphi^{-1}(U))$).
(2) Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$. Then $(g \circ f)_* = g_* \circ f_*$ in the sense that for any presheaf $\mathcal{F}$ on X we have a natural isomorphism of sheaves $(g \circ f)_*\mathcal{F} \simeq g_*(f_*\mathcal{F})$. This is really just the statement
$$\mathcal{F}((g \circ f)^{-1}(U)) = \mathcal{F}(f^{-1} \circ g^{-1}(U)) = (f_*\mathcal{F})(g^{-1}U).$$
Moreover, this equality is transitive, i.e. $(h \circ g)_*(f_*\mathcal{F}) = h_*((g \circ f)_*\mathcal{F})$ as is easily checked.

1.8. **Stalks.**

Definition 1.9. Let $\mathcal{F}$ be a presheaf on X and $x \in X$. Define the *stalk* of $\mathcal{F}$ at x as
$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U),$$
where the direct limit is formed using the restriction maps.

(1) If X is a $\mathbf{C}$-manifold and $\mathcal{F} = \mathcal{O}_X$ then $\mathcal{O}_{X,x}$ is the set of germs of functions at x.
(2) If $f : X' \to X$ is a covering space and $\mathcal{F} = \Gamma_{X'/X}$ then $\mathcal{F}_x = f^{-1}(x)$. This follows from the fact that f is a local homeomorphism. Since f is a covering map, for small enough $U \ni x$ we have $f^{-1}(U) = \coprod_{s \in f^{-1}(x)} V_s$ for open sets $V_s \subset X'$ which are homeomorphic to U via f. Then an element $s \in f^{-1}(x)$ is just the map $s : U \to V_s$ given by f^{-1}. Evidently $f \circ s(x) = x$ and this condition uniquely determines any map $s : U \to f^{-1}(U)$ when U is small enough (which is all we care about since we are taking a direct limit).