1. Types of maps

Let $f : X \to Y$ be a map of schemes.

(1) f is **quasi-compact** if equivalently
   (a) There exists an open affine cover U_i of Y such that $f^{-1}(U_i)$ is quasi-compact.
   (b) For every open quasi-compact $U \subseteq Y$ we have $f^{-1}(U)$ quasi-compact.

(2) f is **locally of finite type** if equivalently
   (a) There exists an open affine cover U_i of Y and an open affine cover V_{ij} of $f^{-1}(U_i)$ such that $\mathcal{O}_X(V_{ij})$ is a finitely generated $\mathcal{O}_Y(U_i)$-algebra.
   (b) For every open affine $U \subseteq Y$ and any open affine $V \subseteq f^{-1}(U)$, $\mathcal{O}_X(V)$ is a finitely generated $\mathcal{O}_Y(U)$-algebra.

(3) f is **of finite type** if it is locally of finite type and quasi-compact.

(4) f is **affine** if equivalently
   (a) There exists an open affine cover U_i of Y such that $f^{-1}(U_i)$ is affine.
   (b) For every open affine $U \subseteq Y$, $f^{-1}(U)$ is affine.

(5) f is **finite** if
   (a) There exists an open affine cover U_i of Y with $f^{-1}(U_i) = V_i$ affine and $\mathcal{O}_X(V_i)$ is a finite $\mathcal{O}_Y(U_i)$-module.
   (b) For any open affine $U \subseteq Y$, $f^{-1}(U) = V$ is affine and $\mathcal{O}_X(V)$ is a finite $\mathcal{O}_Y(U)$-module.

The equivalence of each part of these definitions follows easily from "Nike's Trick."

Definition 1.1. Let X be a scheme. We say X is *integral* if $\mathcal{O}_X(U)$ is a domain for every open $U \subseteq X$.

Definition 1.2. Let X be a scheme. We say X is *irreducible* if the underlying topological space $|X|$ of X is irreducible (*i.e.* can not be written as the union of two proper closed subsets).

Definition 1.3. Let X be a scheme. Then X is *reduced* if $\mathcal{O}_X(U)$ is a reduced ring for all open $U \subseteq X$, or, equivalently, if $\mathcal{O}_{X,x}$ is reduced for all points $x \in X$.

Theorem 1.4. *Let X be a scheme. Then X is integral if and only if it is irreducible and reduced.*

Proof. Clearly X integral implies that X is reduced. Suppose that X is reducible. Then we can write $X = Z_1 \cup Z_2$ with Z_i proper and closed subsets of X. Let $U_i = Z_i - X_i$. Then the U_i are open and disjoint, so by the sheaf axiom we have $\mathcal{O}_X(U_1 \cup U_2) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$. Now as U_1, U_2 are nonempty the rings $\mathcal{O}_X(U_i)$ are nonzero (as they map to the local ring $\mathcal{O}_{X,x} \neq 0$ for any $x \in U_i$). It follows that since $\mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$ is not a domain, we must have had X irreducible.

Conversely, suppose that X is reduced and irreducible. Suppose that $a_1, a_2 \in \mathcal{O}_X(U)$ satisfy $a_1 a_2 = 0$. Put $Y_i = \{x \in U \mid a_i(x) = 0 \text{ in } k(x)\}$. Evidently $Y_1 \cup Y_2 = U$. We claim that Y_i is closed. Indeed, the complement $U - Y_i = \{x \in U \mid a_i(x) \neq 0 \text{ in } k(x)\}$. But for any $x \in U - Y_i$ the image of a_i in $\mathcal{O}_{U,x}$ is not in the maximal ideal and is hence a unit, so there exists an open V_x containing x with a_i a unit in $\mathcal{O}_U(V_x)$, so that for every $y \in V_x$ we have $a_i(y) \neq 0 \in k(y)$, that is $V_x \subseteq U - Y_i$. Hence $U - Y_i$ is open. Now as U is irreducible (since X is) we have $Y_1 = U$, say. It follows that $a_1(x) = 0$ for all $x \in U$. Thus for any affine $V = \operatorname{Spec} R \subseteq U$ we have $a_1\big|_V = 0$ since $a_1\big|_V$ is contained in every prime ideal of R is then nilpotent, and R is reduced. It follows that $a_1 = 0$ and $\mathcal{O}_X(U)$ is reduced. ∎

Definition 1.5. A scheme X is *locally Noetherian* if there exists an open affine cover $\{U_i\}$ of X with each $\mathcal{O}_X(U_i)$ Noetherian.

The equivalence of this definition with the corresponding "string definition" (that is, for any open affine $U \subseteq X$ $\mathcal{O}_X(U)$ is Noetherian) follows from "Nike's Trick" and

Theorem 1.6. *If A is a ring and $(f_1, \dots f_n)A = A$ with A_{f_i} Noetherian then A is Noetherian.*

Proof. Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$ be an ascending chain of ideals in A. Then since each A_{f_i} is Noetherian, we can omit some initial terms of this sequence so that $\mathfrak{a}_i A_{f_j} = \mathfrak{a}_{i+1} A_{f_j}$ for all i, j. But as the f_j generate the unit ideal, for any prime ideal $\mathfrak{p}$ of A there exists some $f_j \notin \mathfrak{p}$, whence $A_\mathfrak{p}$ is a localization of A_{f_j} so that $\mathfrak{a}_i A_\mathfrak{p} = \mathfrak{a}_{i+1} A_\mathfrak{p}$ for all i. As this holds for all $\mathfrak{p}$ we see that $\mathfrak{a}_i = \mathfrak{a}_{i+1}$ for all i so that the chain stabilizes. Hence A is Noetherian. ■

Definition 1.7. A scheme X is *Noetherian* if it is locally Noetherian and quasi-compact.

Example 1.8. Let $R = \prod_{i=1}^{\infty} \mathbf{F}_2$. Then every element of R is idempotent, so that the localization of R at any maximal ideal is $\mathbf{F}_2$ (since any local ring in which every element is idempotent is a field) and is hence Noetherian. But $\operatorname{Spec} R$ is decidedly not Noetherian.

Definition 1.9. A subspace Y of X is an *irreducible component* if it is a maximal closed and irreducible subset.

A fact (which appears on the homework) is that and irreducible closed subset of a scheme X lies in some irreducible component. (This is equivalent to the fact that any prime of a ring contains a minimal prime, which follows from Zorn's lemma). Moreover, we have a bijection

$$\{\text{Irreducible components of } X\} \leftrightarrow \{\xi \in X \mid \mathcal{O}_{X,\xi} \text{ is 0-dimensional}\}$$

given by

$$\overline{\{\xi\}} \leftarrow \xi.$$